M E T U
Department of Mathematics

	Field Extensions and Galois Theory **Midterm I**	
Code : *Math 368* Acad. Year : *2017-2018* Semester : *Spring* Instructor : *Karayayla* Date : *28.03.2018* Time : *17.40* Duration : *120 minutes*	Last Name : Name : Student No : Department : Signature : **7** Questions on **5** Pages SHOW DETAILED WORK!	

1	2	3	4	5		

1.(16 pts.) For $f \in \mathbb{R}[x]$ of degree 4, let x_1, x_2, x_3 and x_4 be the four roots of f in $\mathbb{C}$. Express the discriminant Δ of f in terms of the roots of f, and show that $\Delta < 0$ if exactly two of the roots of f are real and distinct.

2.(16 pts.) Let $f = x^3 + 2x^2 + 3x + 5$ and α, β, γ be its roots in $\mathbb{C}$. Find a polynomial g of degree 3 whose roots are $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$.

3.(2×9 pts.) a) Assume $f \in F[x]$ is irreducible where F is a field and f does not divide $g \in F[x]$. Show that there are A, $B \in F[x]$ such that $Af + Bg = 1$. (Hint: Consider the ideal $\langle f, g \rangle$ generated by f and g in $F[x]$ and use the fact that $F[x]$ is a PID.)

b) For f, g, A, B as in part (a), show that $B + \langle f \rangle$ is the multiplicative inverse of $g + \langle f \rangle$ in the field $\dfrac{F[x]}{\langle f \rangle}$.

4.(3×6 pts.) Let $F \subset L$ be a field extension and $\alpha \neq 0$, $\alpha \in L$ be algebraic over F.

a) Show that $1/\alpha$ is also algebraic over F.

b) Show that $[F(\alpha) : F] = [F(1/\alpha) : F]$.

c) If $f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$ is the minimal polynomial of α over F, what is the minimal polynomial of $1/\alpha$ over F?

5.(2×7 pts.) a) Show that $f = \frac{1}{3}x^4 + \frac{2}{5}x^3 + \frac{7}{2}x^2 + x + 2 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

b) If $a \in \mathbb{Z}$ is the product of distinct primes, then show that $f = x^n - a \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

6.(Bonus, 10 pts.) For a field extension $F \subset L$, assume that $\alpha \in L$ is algebraic over F such that the degree of its minimal polynomial over F is odd. Prove that $F(\alpha^2) = F(\alpha)$.

7.($10+8$ pts.) a) Let $[F(\alpha):F]=r$ and $[F(\beta):F]=s$ for two elements $\alpha,\beta\in L$ for an extension field L over F. Show that $lcm(r,s)\leq[F(\alpha,\beta):F]\leq rs$ where $lcm(r,s)$ is the least common multiple of r and s.

b) If $\zeta_5=e^{\frac{2\pi i}{5}}$, what is $[\mathbb{Q}(\zeta_5,\sqrt[3]{2}):\mathbb{Q}]$?