Middle East Technical University - Department of Mathematics
Math 473 - Ideals, Varieties and Algorithms
FINAL - January 18, 2018

Name:________________________ ID Number:____________

1. (5×4 pts.) a) For $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in k[x]$ where $a_n \neq 0$, let the homogenization of f be the polynomial $f^h = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 x y^{n-1} + a_0 y^n \in k[x,y]$, that is $f^h = \Sigma_{i=0}^{n} a_i x^i y^{n-i}$. Show that $f(x)$ has a root in k if and only if there exists $(a,b) \neq (0,0) \in k^2$ such that $f^h(a,b) = 0$.
b) If k is not algebraically closed, show that there exists $g \in k[x,y]$ such that $V(g) = \{(0,0)\}$.
c) Generalize part (b) inductively to show that if k is not algebraically closed, then there exists $g \in k[x_1, x_2, ..., x_n]$ such that $V(g) = \{(0,0,...,0)\} \subset k^n$.
d) Show that if k is not algebraically closed and $W = V(g_1, g_2, ..., g_r) \subset k^n$ is a variety, then there exists $f \in k[x_1, x_2, ..., x_n]$ such that $W = V(f)$.
e) Explicitly write down f in part (d) in terms of g_1, g_2,..., g_r in the case $k = \mathbb{R}$.

2. (10+10 pts.) a) Show that if $f, g \in \sqrt{I} \subset k[x_1, ..., x_n]$ where I is an ideal, then there exists $N \in \mathbb{Z}^+$ such that $h^N \in I$ for all $h \in \langle f, g\rangle$.
b) For any infinite field k, show that $J = \langle xy, yz, xz\rangle \subset k[x,y,z]$ is a radical ideal. (Hint: Find $V(J)$ first, and then use $\sqrt{J} \subset I(V(J))$. Division Algorithm will be useful.)

3. (10+10 pts.) a) For an algebraically closed field k, show that if $f \in k[x_1, x_2, ..., x_n]$ is irreducible, then $V(f) \subset k^n$ is an irreducible variety.
b) If k is algebraically closed, using the unique factorization in $k[x_1, x_2, ..., x_n]$, write down the irreducible components of a variety $V(g)$ where $g \in k[x_1, ..., x_n]$ is any non-constant polynomial.

4. (10 pts.) Show that if k is algebraically closed, then $\sqrt{I}$ is the intersection of all maximal ideals containing I for any proper ideal $I \subset k[x_1, x_2, ..., x_n]$.

5. (10 pts.) Let $V \subset W \subset k^n$ be varieties. Show that any irreducible component of V is contained in some irreducible component of W.

6. (10+10 pts.) a) Find the reduced Groebner basis of the ideal $I = \langle x - y^3 z, y^2 + z^3\rangle \in \mathbb{Q}[x,y,z]$ using lex order $x > y > z$.
b) Is $f = xy^2 + x + y^3 z^2 + y^2 z$ in I?