M E T U
Department of Mathematics

	Field Extensions and Galois Theory **FINAL**	

Code : *Math 368* Acad. Year : *2017-2018* Semester : *Spring* Instructor : *Karayayla* Date : *28.05.2018* Time : *9.30* Duration : *150 minutes*	Last Name : Name : Student No : Department : Signature : **7** Questions on **5** Pages SHOW DETAILED WORK!

1	2	3	4	5	6		

1. (8+7+7 pts.) a) Let $F \subset L$ be a finite extension of fields. Write down 4 equivalent conditions for this extension to be a Galois extension.

b) Prove that $F \subset L$ is a Galois extension if and only if for any $\alpha \in L - F$ there exists a $\sigma \in Gal(L/F)$ such that $\sigma(\alpha) \neq \alpha$.

c) Let $F \subset K$ and $K \subset L$ be Galois extensions. Prove that $F \subset L$ is Galois if every $\sigma \in Gal(K/F)$ extends to an automorphism of L.

2. (10+10 pts.) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

a) Show that $Gal(L/\mathbb{Q})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

b) Find all fields K such that $\mathbb{Q} \subset K \subset L$.

3. (4×5 pts.) For each of the following field extensions, determine whether it is a Galois extension or not:

a) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$

b) $\mathbb{Q} \subset$ @(α, β), where α and β are distinct roots of $x^3 + x^2 + 2x + 1$.

c) $F_p(t^p) \subset F_p(t)$ where t is a variable and F_p is finite field with p (prime) elements ($\mathrm{F}_p = \mathbb{Z}/p\mathbb{Z}$).

d)C (t^n) $\subset \mathbb{C}(t)$ where t is a variable and n is a positive integer.

4. (5×4 pts.) Let F be a field extension of $\mathbb{C}$, and assume that $f = x^n - \beta \in F[x]$ is irreducible. Let α be a root of f in some extension field of F.

a) Show that $F(\alpha)$ is a splitting field of f over F.

b) Show that $F \subset F(\alpha)$ is a Galois extension.

c) Show that there exists a $\sigma \in Gal(F(\alpha)/F)$ such that $\sigma(\alpha) = \zeta_n \alpha$ where $\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbb{C} \subset F$ is the primitive nth root of 1.

d) Show that $Gal(F(\alpha)/F) = \langle \sigma \rangle$ and it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (σ as in part (c)).

e) Show that $F \subset K$ is Galois for any intermediate field $F \subset K \subset F(\alpha)$.

5. (10+10 pts.) a) Let H be a subgroup of $Gal(L/F)$ for a field extension $F \subset L$. Show that the fixed field of H in L defined as $L_H = \{\alpha \in L | \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}$ is a subfield of L containing F.
b) Assume that $[L : F] = p^2$ where p is a prime and $L \neq F(\alpha)$ for any $\alpha \in L$. Show that $[F(\beta) : F] = p$ for any $\beta \in L - F$.

6. (Bonus: 10 pts.) For each of the following statements, determine whether it is true or false (No explanation is asked, but two wrong answers will cancel one correct answer):

1) There exists finite extensions $F \subset L$ which have infinitely many intermediate fields. ( )

2) $[L : F] = 120$ where $L = \mathbb{Q}(x_1, ..., x_n)$ (field of rational functions in n variables) and $F = \mathbb{Q}(\sigma_1, ..., \sigma_n)$ where σ_i are elementary symmetric polynomials in $x_1, ..., x_n$. ( )

3) For any finite extension $F \subset L$ there exists an intermediate extension $F \subset K \subset L$, $K \neq L$ such that $K \subset L$ is Galois. ( )

4) If $char(F) \neq 0$ and $F \subset L$ is a separable extension which is not Galois, then there is no extension $L \subset M$ such that $F \subset M$ is Galois. ( )

5) If $f \in F[x]$ is irreducible and $F \subset K$ is a finite extension, then f is irreducible in $K[x]$. ( )

6) If L is a splitting field of $f \in F[x]$ over F and if $F \subset K \subset L$ is an intermediate extension, then L is also a splitting field of f over K. ( )

7) Any finite and normal extension $F \subset L$ is a splitting field. ( )

8) An algebraic extension is a finite extension. ( )

9) For finite extensions $F \subset K \subset L$, if $\alpha \in L - K$ is separable over K, then α is separable over F. ( )

10) For polynomials of degree d ($d \geq 5$) over $\mathbb{Q}$, there is no formula (valid for all degree d polynomials) expressing the roots of the polynomial in terms of the coefficients of the polynomial using the operations of taking radicals (nth root), addition, subtraction, multiplication and division. ( )