M E T U
Department of Mathematics

	Discrete Mathematics Midterm II	
Code : *Math 112* Acad. Year : *2017-2018* Semester : *Spring* Instructor : *Finashin, Emelyanov Karayayla, Onal, Seven* Date : *03.05.2018* Time : *17.40* Duration : *100 minutes*	Last Name : Name : Student No : Department : Signature : 5 Questions on 4 Pages SHOW YOUR WORK!	

1	2	3	4	5		

1.(8+12 pts.) Suppose that a_n is a sequence which satisfies a linear homogeneous recurrence relation with constant coefficients. Assume that the characteristic equation of this recurrence relation is

$$(r-2)^2(r-3) = 0.$$

a) Find the recurrence relation of a_n.

Characteristic Equation: $(r-2)^2 \cdot (r-3) = 0$

$(r^2-4r+4)\cdot(r-3) = 0$

$r^3-4r^2+4r-3r^2+12r-12=0$

Therefore, a_n satisfies $r^3-7r^2+16r-12=0$

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

b) Find an explicit formula for a_n if $a_0 = 7, \quad a_1 = 25, \quad a_2 = 77$.

Roots of the characteristic equation are $r=2$ with multiplicity 2, $r=3$ with multiplicity 1.

General solution of this recurrence relation is then:

$a_n = A\cdot 2^n + B\cdot n\cdot 2^n + C\cdot 3^n$ for constants A, B and C.

$n=0 \Rightarrow a_0 = 7 = A\cdot 2^0 + B\cdot 0\cdot 2^0 + C\cdot 3^0 \Rightarrow 7 = A+C$ (Eq1)

$n=1 \Rightarrow a_1 = 25 = A\cdot 2^1 + B\cdot 1\cdot 2^1 + C\cdot 3^1 \Rightarrow 25 = 2A+2B+3C$ (Eq2)

$n=2 \Rightarrow a_2 = 77 = A\cdot 2^2 + B\cdot 2\cdot 2^2 + C\cdot 3^2 \Rightarrow 77 = 4A+8B+9C$ (Eq3)

Eq2 & Eq3 give

$\left.\begin{array}{l} 100 = 8A+8B+12C \\ 77 = 4A+8B+9C \end{array}\right\} \Rightarrow 100-77 = 4A+3C \Rightarrow \boxed{23 = 4A+3C}$

Using Eq1 now,

$\left.\begin{array}{l} 7 = A+C \\ 23 = 4A+3C \end{array}\right\} \Rightarrow 2 = A, C=5 \Rightarrow 25 = 2\cdot 2 + 2\cdot B + 3\cdot 5 \Rightarrow B=3$

Therefore,

$a_n = 2\cdot 2^n + 3\cdot n\cdot 2^n + 5\cdot 3^n$ for all $n \geq 0, n \in \mathbb{Z}$.

2. (15+5 pts.) Let a_n be the number of strings of length n consisting of the letters A, B, C, D, or E that do *not* contain two consecutive A's. (Note that these strings may not contain some of the 5 letters.)

a) Find a recurrence relation that a_n satisfies.

Case 1, The last letter is not A

$\underbrace{\_\ \_\ \_\ \ldots}_{\text{string of length } n-1 \text{ with no consecutive A's } (a_{n-1} \text{ choices})}\ \_$ ← B, C, D or E (4 choices)

1st 2nd 3rd letter ... n^{th} letter

Case 2: n^{th} letter is A, so $(n-1)$st letter cannot be an A.

$\underbrace{\_\ \_\ \ldots}_{a_{n-2} \text{ choices}}\ \_\ A$

1st 2nd ... $(n-1)$st nth

↳ 4 choices (B, C, D or E)

First $n-2$ letters form a length $n-2$ string with no consecutive A's.

Case 1 and Case 2 are all possible cases, and are disjoint, thus:

$$a_n = 4 \cdot a_{n-1} + 4a_{n-2} \quad \text{for all } n \geq 3$$

(a_n is defined for $n \geq 1$, so $n-2 \geq 1$, $n \geq 3$)

b) Compute a_1, a_2, a_3, a_4 and a_5.

a_1: 1 letter words with no consecutive A's $\Rightarrow a_1 = 5$

$a_2 = 5 \cdot 5 - 1 = 24$ (All 2 letter words minus 1 for excluding AA)

$a_3 = 4a_2 + 4a_1 = 4 \cdot 24 + 4 \cdot 5 = 116$

$a_4 = 4a_3 + 4a_2 = 4 \cdot 116 + 4 \cdot 24 = \cancel{448}\ 464 + 96 = \cancel{544}\ 560$

$a_5 = 4a_4 + 4a_3 = 4 \cdot 5\cancel{44}60 + 4 \cdot 116 = \cancel{2176}\ 2240 + \cancel{448}\ 464 = \cancel{2624}\ 2704$

3. (10+10 pts.) a) A monkey types up a random word of length 3 on a keyboard with 26 letters. Find the probability that the word contains an odd number of vowels (Vowels are A,E,I,O and U).

Number of 3 letter words with 1 Vowel: $\binom{3}{1}\binom{5}{1} \cdot 21 \cdot 2\cancel{4}1 = 15 \cdot 44\cancel{1} = \cancel{}\ 6615$

3 Vowels: $(5 \cdot 5 \cdot 5) = 125$

Probability is $\dfrac{6615 + 125}{26^3} = \dfrac{6740}{26^3}$

26^3 ← all 3 letter words

3.b) Another monkey picks randomly 3 letters from a box containing the 26 letters, without replacement (i.e. once a letter is picked, it is not returned into the box). Find the probability that these letters contain an even number (including 0) of vowels. (e.g. $\{A, B, E\}$ is such a selection).

Number of ways of choosing 3 letters including no vowels : $\binom{21}{3} = \frac{21\cdot 20\cdot 19}{3\cdot 2} = 7\cdot 10\cdot 19 = 1330$

2 vowels : $\binom{5}{2}\binom{21}{1} = \frac{5\cdot 4}{2}\cdot 21 = 210$

Probability : $\frac{1330+210}{\binom{26}{3}} = \frac{1540\cdot 3\cdot 2}{26\cdot 25\cdot 24} = \frac{1540}{2600} = \frac{77}{130}$

4. (10+10 pts.) Suppose that a box contains 3 white and 1 black balls, and a second box contains 2 white and 3 black balls. Suppose also that a person chooses at random one of the boxes, then chooses one of the balls in it.

a) Find the probability that the chosen ball is black.

Probability of choosing 1st box = Probability of choosing the second box = $\frac{1}{2}$

Probability of choosing a black ball from box 1 is $\frac{1}{3+1} = \frac{1}{4}$

Probability of choosing a black ball from 2nd box is $\frac{3}{2+3} = \frac{3}{5}$

Answer : $\underbrace{\frac{1}{2}\cdot\frac{1}{4}} + \underbrace{\frac{1}{2}\cdot\frac{3}{5}} = \frac{1}{8} + \frac{3}{10} = \frac{17}{40}$

Probability that 1st box is chosen and a black ball is chosen from it.

Probability that 2nd box is chosen and a black ball is chosen from it.

b) Suppose that the chosen ball turned out to be black. Find the probability that this black ball has been selected from the first box.

Let E be the event that chosen ball is black, and F be the event that the first box has been chosen in the procedure.

$P(E) = \frac{17}{40}$ from part a. $P(F) = \frac{1}{2}$

$P(E \cap F) = \frac{1}{2}\cdot\frac{1}{3+1} = \frac{1}{8}$

$P(F|E) = \frac{P(F\cap E)}{P(E)} = \frac{\frac{1}{8}}{\frac{17}{40}} = \frac{5}{17}$

Probability that 1st box has been chosen in the procedure given that the chosen ball is black (conditional probability)

5. (10+10 pts.) A die is biased such that its outcomes have the following probabilities:

$$P(1) = P(2) = P(4) = P(5) = P(6), and P(3) = \frac{1}{4}.$$

a) Suppose that this biased die is rolled six times consecutively. What is the probability that in exactly four of these six rolls an even number comes up?

$P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1 \Rightarrow 5\cdot P(1)+\frac{1}{4}=1 \Rightarrow P(1)=3/20$

Let p = Probability of getting an even number when the die is rolled once,
q = " " " " odd " " " " ".

$p=P(2)+P(4)+P(6)=3\frac{3}{20}=9/20$, $q=1-p=\frac{11}{20}$ $\quad (q=P(1)+P(3)+P(5))$

6 rolls, results can be seen as 6 letter strings consisting of E or O (E for even, O for odd). The number of such strings with 4 E's and 2 O's is $\binom{6}{4}$.

Having 1 such result has probability $p^4\cdot q^2=\left(\frac{9}{20}\right)^4\cdot\left(\frac{11}{20}\right)^2$

Since there are $\binom{6}{4}=15$ distinct outcomes with 4 even and 2 odd numbers,

Answer: $\binom{6}{4}\cdot\left(\frac{9}{20}\right)^4\cdot\left(\frac{11}{20}\right)^2=\frac{15\cdot 81\cdot 81\cdot 121}{64\cdot 10^6}$

b) Suppose that together with this biased die a second die, which is fair, is also rolled! (If these 2 dice are rolled once) What is the probability that the two numbers that come up are 3 and 6? (1 outcome is 3, and the other is 6).

Case 1, Biased die's outcome is 3, fair die's outcome is 6

Probability of case 1 is $P(3)\cdot\frac{1}{6}=\frac{1}{4}\cdot\frac{1}{6}=\frac{1}{24}$

Case 2, Biased die's outcome is 6, fair die's outcome is 3

Probability of case 2 is $P(6)\cdot\frac{1}{6}=\frac{3}{20}\cdot\frac{1}{6}=\frac{1}{40}$

Answer: $\frac{1}{24}+\frac{1}{40}=\frac{5+3}{120}=\frac{8}{120}=\frac{1}{15}$