M E T U
Department of Mathematics

Complex Calculus Midterm 2								
Code : *Math 353* Acad. Year : *2017-2018* Semester : *Fall* Date : *December.18.2018* Time : *17:40* Duration : *120 minutes*						Last Name : Name : Student No. : Department : Section : Signature :		
						6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS		
1	2	3	4	5	6	SHOW YOUR WORK		

Question 1 (12 pts) Show that if $f(z)$ is entire and $|f(z)| \leq |e^z|$ for all $z \in \mathbb{C}$, then there exists a constant $c \in \mathbb{C}$ such that $f(z) = ce^z$.

Let $g(z) = \dfrac{f(z)}{e^z}$, since f and e^z are entire and $e^z \neq 0$, $g(z)$ is entire.

$$|g(z)| = \left|\frac{f(z)}{e^z}\right| = \frac{|f(z)|}{|e^z|} \leq 1 \text{ since } |f(z)| \leq |e^z|$$

If $|g(z)| \leq 1$ for all $z \in \mathbb{C}$, and $g(z)$ is entire (analytic on $\mathbb{C}$), then by Liouville's Theorem, $g(z) = c$ for some constant $c \in \mathbb{C}$.

Therefore $g(z) = \dfrac{f(z)}{e^z} = c \Rightarrow f(z) = c \cdot e^z$ for all $z \in \mathbb{C}$

Question 2 (13 pts) Let $f(z)$ be a function which is continuous on a closed and bounded region $R \subset \mathbb{C}$, and suppose that f is analytic and non-constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has an absolute minimum value m on R which occurs on the boundary of R and never in the interior of R.

Existence of m: R is compact (closed and bounded) and $|f(z)|$ is a continuous real valued function on R (since $f(z)$ is continuous on R). Then by Extreme Value Thm, $|f(z)|$ has an absolute min value m, and also abs. max. value M on R.

$0 \leq |f(z)|$ for all z, but m is abs. min $\Rightarrow m \leq |f(z)|$ for all $z \in R$.

$f(z) \neq 0$ on $R \Rightarrow m \neq 0$, so $\boxed{0 < m}$

$0 < m \leq |f(z)|$ for all $z \in R \Leftrightarrow 0 < \dfrac{1}{|f(z)|} \leq \dfrac{1}{m}$

$m = f(z_0)$ for some $z_0 \in R$

$g(z) = \dfrac{1}{f(z)} \Rightarrow g(z)$ is analytic on R and $|g(z)|$ has abs. max at $z_0 \in R$.

$|g(z_0)| = \dfrac{1}{|f(z_0)|} = \dfrac{1}{m}$

$g(z) = \dfrac{1}{f(z)}$ for $z \in R$.

Under those conditions, the theorem says that z_0 cannot be in the interior of R unless $g(z)$ is constant.

$f(z)$ is non-constant, thus $g(z)$ is non-constant.

This shows that z_0 = abs min point of $|f(z)|$ = abs. max pt of $|g(z)|$ is not an interior pt. of R, thus z_0 is a boundary point of R.

Question 3 (6+6+6+7=25 pts) Evaluate the following contour integrals where C is the positively oriented boundary of the given region R

a) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}dz$, $R = \{x+iy \mid |x| \le 2, |y| \le 2\}$.

$f(z) = \dfrac{\cos(z)}{z^2+8}$ is analytic on $\mathbb{C} - \{\pm 2\sqrt{2}i\}$, $\pm 2\sqrt{2}i$ is outside the simple closed contour C.

$f(z)$ is analytic on and inside C, 0 is inside C, then by Cauchy integral formula:

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z}dz = f(0) = \frac{\cos 0}{0^2+8} = \frac{1}{8} \Rightarrow \int_C \frac{\cos z}{z(z^2+8)}dz = 2\pi i \cdot \frac{1}{8} = \frac{\pi i}{4}$$

b) $\displaystyle\int_C \frac{\cosh z}{z^4}dz$, the same R as in part (a).

$f(z) = \cosh(z)$ is entire, hence $f(z)$ is analytic on and inside the simple closed contour C, and $z_0 = 0$ is inside C. Then, by the extension of Cauchy Integral Formula:

$$\frac{3!}{2\pi i}\int_C \frac{\cosh(z)}{z^4}dz = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-0)^{3+1}}dz = f^{(3)}(0) = \sinh(0) = 0 \Rightarrow \int_C \frac{\cosh(z)}{z^4}dz = 0$$

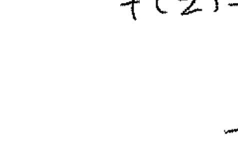
c) $\displaystyle\int_C \frac{z}{(z-1+i)^2}dz$, $R = \{z \in \mathbb{C} \mid |z-2-2i| \le \frac{3}{2}\}$.

$f(z) = \dfrac{z}{(z-(1-i))^2}$ is analytic on $\mathbb{C} - \{1-i\}$. $1-i$ is outside C (simple closed)

Thus $f(z)$ is analytic on and inside C, hence by Cauchy-Goursat Thm

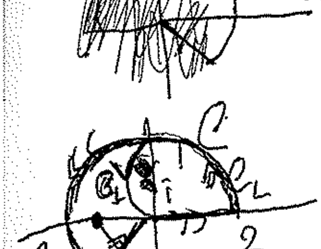
$$\int_C \frac{z}{(z-1+i)^2}dz = \int_C f(z)dz = 0.$$

d) $\displaystyle\int_C \frac{1}{(z^2+1)(z^2-2z-3)}dz$, $R = \{z \in \mathbb{C} \mid |z| \le 2, 0 \le arg(z) \le 5\pi/4\}$.



$$\frac{1}{(z^2+1)(z^2-2z-3)} = \frac{1}{(z-i)(z+i)(z-3)(z+1)} \rightarrow$$ not analytic on only $z = \pm i$ and $z=3, z=-1$

$z_0 = i$ and $z_1 = -1$ are inside C, $-i$ and 3 are outside C



$$\int_C \frac{1}{(z^2+1)(z^2-2z-3)}dz = \int_{C_1+C_2} \frac{1}{(z-i)(z+i)(z^2-2z-3)}dz + \int_{C_3-C_1} \frac{1}{(z+1)(z-3)(z^2+1)}dz$$

i inside the contour

-1 inside the contour

Then by Cauchy Integral Formula on $C_1 + C_2$ and $C_3 - C_1$ the result is:

$$2\pi i\left(\frac{1}{(i+i)(i^2-2i-3)} + \frac{1}{(-1-3)((-1)^2+1)}\right) = \frac{4i-8}{4(4-8i)}\pi$$

$$= 2\pi i\left(\frac{1}{4-8i} + \frac{1}{-8}\right)$$

$C = (C_1 + C_2) + (C_3 - C_1)$

simple closed, positively oriented.

Question 4(7+8=15 pts) Let $f(z) = \log z = \ln r + i\theta$, where $-\pi/4 < \theta < 7\pi/4$, and let C be the part of the graph of the polar equation $r = 1 + \sin(2\theta/3)$ where $0 \le \theta \le 3\pi/2$. C is given a direction such that its initial point is 1.

a) Express $\int_C f(z)dz$ as a definite integral using a parametrization of C.

Use θ as a parameter. $z = re^{i\theta}$, so $dz = e^{i\theta}dr + rie^{i\theta}d\theta$. $dr = \frac{2}{3}\cos(2\theta/3)d\theta$.

Hence,

$$\int_C f(z)dz = \int_0^{3\pi/2} \left(\ln\left(1+\sin(2\theta/3)\right) + i\theta\right)\left(e^{i\theta}\cdot\frac{2}{3}\cos\left(\frac{2\theta}{3}\right) + \left(1+\sin\left(\frac{2\theta}{3}\right)\right)ie^{i\theta}\right)d\theta$$

b) Evaluate $\int_C f(z)dz$ without using a parametrization of the given contour C.

The contour C lies in a region which is simply connected and in which f is analytic $\left(\frac{-\pi}{4} < \theta < \frac{7\pi}{4}\right)$. So it must have an antiderivative $F(z)$.

Actually, for $F(z) = z\log z - z$, we have $F'(z) = f(z)$.

$$\int_C f(z)dz = F(1) - F(-i) = (1\cdot\underbrace{\cancel{\ln 1}}_{0} - 1) - \left(-i\cdot\left(\underbrace{\cancel{\ln 1}}_{0} + i\frac{3\pi}{2}\right) - (-i)\right)$$

$$= -1 - \frac{3\pi}{2} - i$$

Question 5 (15 pts) Find the Taylor series expansion of $f(z) = \dfrac{\cos(z)}{z}$ around the point $z_0 = \pi$ up to and including the $(z-\pi)^3$ term. What is the radius of convergence of this series? (i.e. find the largest $R > 0$ such that $f(z)$ is equal to this Taylor series on the open disk $|z - \pi| < R$.)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(z-\pi)^n.$$

$$f(\pi) = \frac{\cos(\pi)}{\pi} = \frac{-1}{\pi}$$

$$f'(z) = \frac{-\sin(z)\cdot z - \cos(z)}{z^2} \Rightarrow f'(\pi) = \frac{1}{\pi^2}$$

$$f''(z) = \frac{\left(-\cos(z)\cdot z - \cancel{\sin(z)} + \cancel{\sin(z)}\right)z^2 - 2z\cdot\left(-\sin(z)\cdot z - \cos(z)\right)}{z^4}$$

$$= \frac{-\cos(z)}{z} + \frac{2\sin(z)}{z^2} + \frac{2\cos(z)}{z^3} \Rightarrow f''(\pi) = +\frac{1}{\pi} - \frac{2}{\pi^3}$$

$$f'''(z) = \frac{\sin(z)\cdot z + \cos(z)}{z^2} + \frac{2\cos(z)\cdot z^2 - 2z\cdot(2\sin(z))}{z^4} + \frac{-2\sin(z)\cdot z^3 - 3z^2\cdot(2\cos(z))}{z^6}$$

$$\Rightarrow f'''(\pi) = \frac{-1}{\pi^2} + \frac{-2}{\pi^2} + \frac{6}{\pi^4} = \frac{-3}{\pi^2} + \frac{6}{\pi^4}$$

So, $f(z) = \frac{-1}{\pi} + \frac{1}{\pi^2}(z-\pi) + \frac{1}{2!}\left(\frac{1}{\pi} - \frac{2}{\pi^3}\right)(z-\pi)^2 + \frac{1}{3!}\left(\frac{-3}{\pi^2} + \frac{6}{\pi^4}\right)(z-\pi)^3 + \cdots$

The radius of convergence is π, since the nearest singularity is at $z=0$.

Question 6 (6+7+7=20 pts) Find the Laurent series expansions of the function $f(z) = \dfrac{z^2}{(z-2)(z+3)}$ centered at $z_0 = 0$, where the representation is valid;

(a) in the region $0 \leq |z| < 2$,

$$(z-2)(z+3) = z^2 + z - 6 \Rightarrow \frac{z^2}{(z-2)(z+3)} = 1 + \frac{-z+6}{(z-2)(z+3)}$$

$$\frac{-z+6}{(z-2)(z+3)} = \frac{A}{z-2} + \frac{B}{z+3} \Rightarrow -z+6 = A(z+3) + B(z-2)$$

$$\left.\begin{array}{l} A+B=-1 \\ +3A-2B=6 \end{array}\right\} A = \frac{4}{5}, \quad B = \frac{-9}{5}$$

$$\left.\begin{array}{l} \dfrac{1}{z-2} = \dfrac{-1}{2}\cdot\dfrac{1}{1-\frac{z}{2}} = \dfrac{-1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \text{ for } |z|<2. \\ \dfrac{1}{z+3} = \dfrac{1}{3}\left(\dfrac{1}{1-\left(\frac{-z}{3}\right)}\right) = \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n \text{ for } |z|<3 \end{array}\right\} \text{both are valid for } |z|<2.$$

$$\Rightarrow f(z) = 1 + \frac{4}{5}\left(\frac{-1}{2}\right)\sum_{n=0}^{\infty}\left(\frac{+z}{2}\right)^n + \left(\frac{-9}{5}\right)\cdot\frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n.$$

(b) in the region $2 < |z| < 3$,

We can use the same expansion for $\frac{1}{z+3}$ as in part (a), since $|z|<3$, but we must find a new expansion for $\frac{1}{z-2}$ since $|z|>2$.

$$\frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \text{ for } |z|>2.$$

$$\Rightarrow f(z) = 1 + \frac{4}{5}\cdot\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \left(\frac{-9}{5}\right)\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n.$$

(c) in the region $3 < |z|$.

Here, we must also find a new expansion for $\frac{1}{z+3}$ since $|z|>3$.

$$\frac{1}{z+3} = \frac{1}{z}\cdot\left(\frac{1}{1-\left(\frac{-3}{z}\right)}\right) = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{-3}{z}\right)^n.$$

$$\Rightarrow f(z) = 1 + \frac{4}{5}\cdot\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \left(\frac{-9}{5}\right)\cdot\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{-3}{z}\right)^n.$$