M E T U
Department of Mathematics

	Complex Calculus MidTerm 1	
Code : *Math 353* Acad. Year : *2017-2018* Semester : *Fall* Date : *November.13.2017* Time : *17:40* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		**SHOW YOUR WORK**	

Question 1 (10+10=20 pts) a) Show that for any complex numbers z and w, $Re(z) > 0$ and $Re(w) > 0$ implies that $Arg(zw) = Arg(z) + Arg(w)$. ($Re(z)$: real part of $z \in \mathbb{C}$. $Arg(z)$: principal argument of $z \in \mathbb{C}$.)

$$Re(z) > 0 \Rightarrow \frac{-\pi}{2} < Arg(z) < \frac{\pi}{2}$$

$$Re(w) > 0 \Rightarrow \frac{-\pi}{2} < Arg(w) < \pi/2 .$$

Therefore, $Arg(zw)$, which is the member of $arg(zw) = arg(z) + arg(w)$ such that $-\pi < Arg(zw) \leq \pi$ must coincide with $Arg(z) + Arg(w)$.

b) Assume that $\lim_{z\to z_0} f(z) = 0$ and there exists $M > 0$ ($M \in \mathbb{R}$) such that $|g(z)| < M$ for all z in some neighborhood of z_0. Show that $\lim_{z\to z_0} f(z)g(z) = 0$.

Suppose that $\varepsilon > 0$ is given. Since $M > 0$, we have $\varepsilon/M > 0$. By definition of limit, there exists $\delta_1 > 0$ such that

$0 < |z - z_0| < \delta_1$ implies $|f(z)| < \varepsilon/M$.

By the given condition on g, there exists $\delta_2 > 0$ such that

$|z - z_0| < \delta_2$ implies $|g(z)| < M$

Take $\delta = \min(\delta_1, \delta_2)$. Then,

$$0 < |z - z_0| < \delta \Longrightarrow |f(z)g(z)| = |f(z)||g(z)| < \frac{\varepsilon}{M} \cdot M = \varepsilon .$$

Question 2 (6+6+6=18 pts) Let $f(z) = 3x^3 - 2y^3 + x^2 + y^2 + i(3x^2y + 2xy^2)$ where $z = x + iy$ $(x, y \in \mathbb{R})$.

a) Find all points $z \in \mathbb{C}$ such that $f'(z)$ exists.

$f(z) = u(x,y) + i\,v(x,y)$ where $u(x,y) = 3x^3 - 2y^3 + x^2 + y^2$

$v(x,y) = 3x^2y + 2xy^2$

• u and v are polynomials, thus u_x, u_y, v_x and v_y are continuous.

Then $f'(z)$ exists at $z = x+iy$ if Cauchy-Riemann Equations hold:

$u_x = v_y \Rightarrow 9x^2 + 2x = 3x^2 + 4xy \Rightarrow 6x^2 - 4xy + 2x = 0$

$2x(3x - 2y + 1) = 0$

$\boxed{x = 0 \vee 3x - 2y + 1 = 0}$

$u_y = -v_x \Rightarrow -6y^2 + 2y = -(6xy + 2y^2)$

$6xy - 4y^2 + 2y = 0$

$2y(3x - 2y + 1) = 0$

$\boxed{y = 0 \vee 2x - 3y + 1 = 0}$

Solution set:

$S = \{(x,y) \in \mathbb{R}^2 \mid 3x - 2y + 1 = 0\} \cup \{(0,0)\}$

$\therefore f'(z)$ exists iff $z = x + iy$ where $(x,y) \in S$

b) Find all points $z \in \mathbb{C}$ such that $f'(z) = 0$.

$f'(z) = 0 \Leftrightarrow f'(z)$ exists $\wedge$ $f'(z) = u_x + i v_x = 0$

$u_x + i v_x = 0 \Leftrightarrow \left.\begin{matrix} u_x = 0 \\ v_x = 0 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} 9x^2 + 2x = 0 \\ 6xy + 2y^2 = 0 \end{matrix}\right\} \Leftrightarrow \begin{matrix} (x = 0 \vee x = -\frac{2}{9}) \\ \wedge (y = 0 \vee y = -3x) \end{matrix}$

solution set:

$\{(0,0), (-\frac{2}{9}, 0), (-\frac{2}{9}, \frac{2}{3})\}$

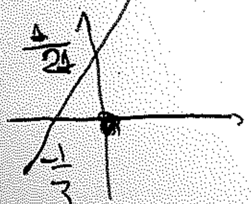
Among those 3 points, only $(0,0) \in S$ among the 3 solutions)

($f'(z)$ exists for only $z = 0$

$\boxed{f'(z) = 0 \Leftrightarrow z = 0}$

c) If it exists, determine the largest set $D \subset \mathbb{C}$ such that $f(z)$ is analytic on D.

$f'(z)$ exists for only $z = 0$ and for z on the line $3x - 2y + 1 = 0$



This set contains no open disk.

There is no z_0 such that $f'(z)$ exists for all z in an open neighborhood $B_\varepsilon(z_0) = \{z \mid |z - z_0| < \varepsilon\}$ of z_0.

Thus $f(z)$ is not analytic on any point z_0.

Thus, $D = \emptyset$

Question 3 (14 pts) Let $V(x,y) = 5x^4y - 10x^2y^3 + y^5 + 2xy$. Find a function $U(x,y)$ (if it exists) such that $F(z) = U(x,y) + iV(x,y)$ is an entire function, where $z = x + iy$ such that $x, y \in \mathbb{R}$. $F(z) = U(x,y) + iV(x,y)$ is entire if U and V have continuous first order partials for all $(x,y) \in \mathbb{R}^2$ and satisfy Cauchy-Riemann Equations.

$U_x(x,y) = V_y(x,y) = 5x^4 - 30x^2y^2 + 5y^4 + 2x$

Then $U(x,y) = \int 5x^4 - 30x^2y^2 + 5y^4 + 2x \, dx = x^5 - 10x^3y^2 + 5y^4x + x^2 + g(y)$ for some function $g(y)$

$U_y = -V_x \Rightarrow \frac{\partial}{\partial y}(x^5 - 10x^3y^2 + 5y^4x + x^2 + g(y)) = -V_x$

$\Rightarrow -20x^3y + 20y^3x + g'(y) = -(20x^3y - 20xy^3 + 2y) \Rightarrow g'(y) = -2y \Rightarrow g(y) = -y^2 + c$ for some $c \in \mathbb{R}$

Therefore,

$U(x,y) = x^5 - 10x^3y^2 + 5y^4x + x^2 - y^2 + c$, for some $c \in \mathbb{R}$ $\Leftrightarrow$ Cauchy-Riemann Eq. hold.

Note that $U(x,y)$ and $V(x,y)$ have continuous first order partial derivatives for all $(x,y) \in \mathbb{R}^2$.

$\therefore F(z) = U(x,y) + iV(x,y)$ is analytic for all (x,y), thus $F(z)$ is entire.

Question 4 (8+10=18 pts) (a) Prove the identities $\cosh^2 x - \sinh^2 x = 1$ and $\sinh x + \cosh x = e^x$ for $x \in \mathbb{R}$ by using the definitions of the functions directly.

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$ (By definition)

$\sinh(x) + \cosh(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x$ for all $x \in \mathbb{R}$

$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(e^{2x} + 2e^xe^{-x} + e^{-2x} - (e^{2x} - 2e^xe^{-x} + e^{-2x})\right)$

$= \frac{1}{4}(2e^0 - (-2e^0)) = \frac{4}{4} = 1$ for all $x \in \mathbb{R}$

(b) Deduce that $\cosh^2 z - \sinh^2 z = 1$ and $\sinh z + \cosh z = e^z$ for all $z \in \mathbb{C}$ from part (a) by using the theorem on uniqueness of analytic extensions (analytic continuation).

If $f(z)$ and $g(z)$ are analytic on a domain D (open and connected subset of $\mathbb{C}$), and if $f(z) = g(z)$ on a line segment or an open subset in D, then $f(z) = g(z)$ for all $z \in D$. (By the theorem)

For $\cosh^2 z - \sinh^2 z = 1$ identity; it is proved by taking $D = \mathbb{C}$, $f(z) = \cosh^2(z) - \sinh^2(z)$, $g(z) = 1$ (constant func.) (which are both entire), and noting $f(z) = g(z)$ on $\mathbb{R} \subseteq \mathbb{C}$ (in particular on the line segment $[0,1]$).

Similarly, for $\cosh(z) + \sinh(z) = 1$ identity:

$f(z) = \cosh(z) + \sinh(z)$ is entire (analytic on $D = \mathbb{C}$), $g(z) = 1$ is entire.

$f(z) = g(z)$ on $\mathbb{R} \subseteq \mathbb{C}$ $(= D)$

Hence $f(z) = g(z)$ on the line segment $[0,1] \subseteq \mathbb{C}$, therefore, by the stated theorem, $f(z) = g(z)$ for all $z \in \mathbb{C}$.

Question 5 (13 pts) Suppose that $f(z)$ is analytic on a domain (an open and connected set) $D \subseteq \mathbb{C}$ and for all $z \in D$, we have

$$Re(f(z)) = 2Im(f(z)).$$

Prove that $f(z)$ must be constant on D.

Suppose that $f = u + iv$ where u, v are real. Then $u = 2v$ for $z \in D$. Since f is analytic on D, Cauchy–Riemann equations must be satisfied.

$$u_x = v_y, \quad u_y = -v_x.$$

But then $u_x = u_y/2$, $u_y = -u_x/2$, implying $u_x = u_y = 0$. Hence, also $v_x = v_y = 0$. Therefore f must be constant. (since D is connected)

Question 6 (17 pts) Let $f(z)$ be the linear fractional transformation such that $f(0) = 1$, $f(1) = i$ and $f(i) = 0$. What is the image of the unit circle under f?

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longleftarrow & 1 \\ 1 & \longrightarrow & 1 & \longleftarrow & i \\ i & \longrightarrow & \infty & \longleftarrow & 0 \\ & g & & h & \end{array}$$

Let g and h be the FLT's taking $(0, 1, i)$ to $(0, 1, \infty)$ and $(1, i, 0)$ to $(0, 1, \infty)$ respectively. Then $f = h^{-1} \circ g$.

$g(z) = \dfrac{z}{z-i} \cdot \dfrac{1-i}{1}$, corresponding to the matrix $\begin{bmatrix} -i+1 & 0 \\ 1 & -i \end{bmatrix}$

$h(z) = \dfrac{z-1}{z} \cdot \dfrac{i}{i-1}$, corresponding to the matrix $\begin{bmatrix} i & -i \\ i-1 & 0 \end{bmatrix}$

Therefore $h^{-1}(z)$ corr. to $\begin{bmatrix} 0 & i \\ 1-i & i \end{bmatrix}$

So f corr. to $\begin{bmatrix} 0 & i \\ 1-i & i \end{bmatrix} \begin{bmatrix} -i+1 & 0 \\ 1 & -i \end{bmatrix} = \begin{bmatrix} i & 1 \\ \cancel{}\ -i & 1 \end{bmatrix}$

so $f(z) = \dfrac{iz + 1}{-\cancel{}iz + 1}$

f should take circles and lines to circles and lines. Check the image of one more point on the unit circle: $f(-i) = \infty$. So the image of the unit circle must be the imaginary axis.