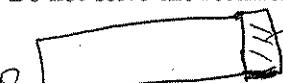


**M E T U**  
**Department of Mathematics**

Discrete Mathematics FINAL EXAM	
Code : Math 112	Last Name :
Acad. Year : 2017-2018	Name : Student No:
Semester : Spring	Department :
Instructor : Emelyanov, Finashin, Karayayla, Önal, Seven	Signature :
Date : 24.05.2018	5 Questions on 4 Pages
Time : 09.30	Total 100 Points
Duration : 120 minutes	

**S O L U T I O N      |      K E Y**

**Q1. (10 + 10 pts.)** a) A  $2 \times n$  rectangle is covered by smaller rectangles of dimensions  $2 \times 1$  and  $2 \times 2$ . If the small pieces of rectangles can be red or yellow in color, find a recursion relation for  $a_n$  where  $a_n$  is the number of ways of covering the large rectangle with these small rectangles. (Give enough number of initial values. Do not solve the recurrence relation.)

Case 1:  red or yellow (2 choices)

$n-1$

$2 \cdot a_{n-1}$  ways for Case 1.

Case 2.

 red or yellow (2 choices)

$a_{n-2}$  ways

$\Rightarrow 2 \cdot a_{n-2}$  ways  
in Case 2

These are all cases.

Therefore  $a_n = 2a_{n-1} + 2a_{n-2} + 4a_{n-2}$  for  $n \geq 3$

$$a_n = 2a_{n-1} + 6a_{n-2} \quad \text{for } n \geq 3;$$

$$\begin{array}{l} \square - \text{red or} \\ \square - \text{yellow} \end{array} \quad \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} \quad \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} \quad a_1 = 2, a_2 = 2 + 4 + 4 = 10 \Rightarrow a_2 = 10$$

b) Letters of the word CLASSROOM are shuffled randomly. Find the probability that two identical letters appear consecutively. 9 letters, 2 are S, 2 are O, others are distinct.

# of permutations including **[SS]** as a block:  $(7+1)! / 2! = \frac{8!}{2!} = 4 \cdot 7!$

# of " " " " " =  $8! / 2! = 2 \cdot 7! = 2 \cdot 7! = 4 \cdot 7!$  identical.

# of permutations including both **[OO]** and **[SS]**:  $(5+2)! = 7! = 7!$

# of permutations including 2 consecutive identical letters:  $4 \cdot 7! + 4 \cdot 7! - 7! = 7 \cdot 7!$

# of all permutations:  $\frac{9!}{2! \cdot 2!} = \frac{9 \cdot 8 \cdot 7!}{2 \cdot 2} = 18 \cdot 7! = 7 \cdot 7!$

Probability that there are consecutive identical letters:  $\frac{7 \cdot 7!}{18 \cdot 7!} = \frac{7}{18}$

Q2. (10 + 10 pts.) a) Let  $A \subseteq X = \{1, 2, \dots, 20\}$  where  $|A| \geq 12$ . Show that there are two elements  $a, b \in A$  such that  $a + b = 19$ .

Let  $A = \{a_1, a_2, \dots, a_n\}$  }  $n = |A| \Rightarrow a_1, a_2, \dots, a_n$  distinct

Let  $b_i = 19 - a_i$  for  $i = 1, 2, \dots, n \Rightarrow b_1, b_2, \dots, b_n$  are distinct

$1 \leq a_i \leq 20 \Rightarrow -20 \leq -a_i \leq -1 \Rightarrow 19 - 20 \leq 19 - a_i \leq 18 \Rightarrow -1 \leq b_i \leq 18$

$a_1, a_2, \dots, a_n, b_1, \dots, b_n$  are  $2n$  integers between  $-1$  and  $20$

There are  $20 - (-1) + 1 = 22$  integers from  $-1$  to  $20$ , and we have  $2n = 2|A|$

Then at least 2 of  $a_1, \dots, a_n, b_1, \dots, b_n$  are equal by Pigeonhole Principle.

≥ 2 · 12

= 24

numbers

$a_1, \dots, a_n$  are distinct }  $\Rightarrow a_i = b_j$  for some  $i$  and  $j$ . Thus,

$b_1, \dots, b_n$  are distinct

$$\begin{aligned} a_i &= 19 - a_j \\ a_i + a_j &= 19 \end{aligned}$$

since

they

are

integers

$$(a_i - a_j) = 19/2$$

b) Let  $B \subseteq Y = \{1, 2, \dots, 112\}$  where  $|B| = 85$ . Show that there are at least 4 consecutive numbers in  $B$ .

$$|Y - B| = 112 - 85 = 27$$

1 2 3 0 ... 0 .. 0 .. 0 0 .. 0 .. 0

Order elements of  $Y$  1 to 112, mark the ones not in  $B$  by 0 and ones in  $B$  by \*. 27 0's separate the \*'s into 28 pieces of consecutive integers. There are 85 \*'s in 28 pieces (groups)

(groups)

$\lceil \frac{85}{28} \rceil = 4 \Rightarrow$  By Pigeonhole Principle, 1 of the pieces contains at least 4 \*, so there are at least 4 consecutive integers in  $B$ .

Q3. (10 + 10 pts.) a) A student reads a book of 100 pages in 9 days by reading at least one page everyday. Assuming that a whole number (integer) of pages are read each day, show that there are two consecutive days in which the student reads at least 21 pages in total.

Assume that there are no consecutive days in which a total of 21 pages are read.

Then in any consecutive 2 days, at most 20 pages are read.

Let  $y_1 = \text{number of pages read in day } 1$ . Then  $y_1 + y_2 + \dots + y_9 = 100$ .

$$\left. \begin{array}{l} y_1 = y_1 + y_2 \\ y_2 = y_2 + y_3 \\ \vdots \\ y_9 = y_9 + y_1 \end{array} \right\} \Rightarrow y_1 \leq 20 \quad (\text{by the above assumption})$$

$$\left. \begin{array}{l} y_2 \leq 20 \\ y_3 \leq 20 \\ \vdots \\ y_9 \leq 20 \end{array} \right\} \quad y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 = 100$$

$$y_1 + y_2 + y_3 + y_4 \leq 80$$

$$100 - y_9 \leq 80 \Rightarrow y_9 \geq 20$$

It is given that  $y_i \geq 1$  for all  $i$ , then  $y_8 + y_9 \geq 1 + 20 = 21$  (contradict with the assumption).

So, such a consecutive 2 days in which at least 21 pages are read exists. (in total)

(with at least 2 vertices)

Q3.b) Prove that any connected graph which has no loops and no multiple edges has a pair of vertices  $u$  and  $v$  such that  $\deg(u) = \deg(v)$  (degrees of the two vertices are equal) and has  $n$  edges.

Let  $G$  be such a graph with  $n$  edges.  $G$  is connected  $\Rightarrow$   $\deg(v) \geq 1$  for all vertices  $v$  of  $G$ .

Any  $v$  can be connected by at most one edge (since no multiple edges) to the  $n-1$  other edges (no loops exist,  $v$  cannot be joined to  $v$  itself).

Thus  $\deg(v) \leq n-1$  for all vertices  $v$  of  $G$ .

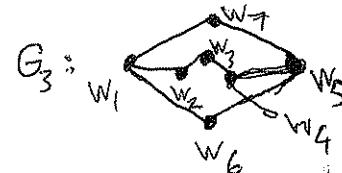
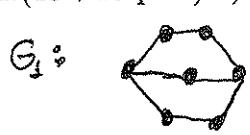
We obtain,  $1 \leq \deg(v) \leq n-1$  for all vertices of  $G$ .

$d_i = \deg(v_i) \Rightarrow d_1, d_2, \dots, d_n$  are  $n$  numbers between 1 and  $n-1$ .

Then by Pigeonhole Principle,  $d_i = d_j$  for some  $i \neq j$ ,  $\deg(v_i) = \deg(v_j)$

$v_i \neq v_j$  distinct vertices.

Q4.(10+10 pts.) a) For each pair of the 3 graphs below, show whether they are isomorphic or not.



$G_1$  has no 4-cycle  $C_4$  as a subgraph, but  $G_2$  and  $G_3$  both have 4 cycles, so  $G_1$  and  $G_2$  are NOT isomorphic.  $G_1$  and  $G_3$  are NOT isomorphic.

If we number the vertices of  $G_2$  and  $G_3$  as  $v_1, v_2, \dots, v_7$  and  $w_1, w_2, \dots, w_7$  as shown, then the adjacency matrices or lists of adjacent vertices are the same for  $G_2$  and  $G_3$ .

$$\text{so } f(v_i) = w_i$$

is an isomorphism between  $G_2$  and  $G_3$

b) Sketch at least 5 non-isomorphic trees with 6 vertices. (No explanation is asked, only sketch the trees.)



$$\text{degrees: } (2, 2, 2, 2, 1, 1)$$



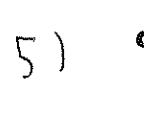
$$(5, 1, 1, 1, 1, 1)$$



$$\rightarrow \text{degrees: } (4, 2, 1, 1, 1, 1)$$



$$\rightarrow \text{degrees: } (3, 3, 1, 1, 1, 1)$$



$$\rightarrow \text{degrees: } (3, 2, 2, 1, 1, 1) \leftarrow 6)$$

(No 2 are isomorphic trees)

Vertex	Adjacent Vertices	Same diagram with $v_i$ replaced by $w_i$	
		adj. vert.	adj. vert.
$v_1$	$v_2, v_6, v_7$	$w_1, w_2, w_6, w_7$	✓
$v_2$	$v_1, v_3$	$w_2, w_3, w_4, w_5$	✓
$v_3$	$v_2, v_4$	$w_3, w_4, w_5, w_6$	✓
$v_4$	$v_3, v_5$	$w_4, w_5, w_6, w_7$	✓
$v_5$	$v_4, v_6, v_7$	$w_5, w_6, w_7, w_1$	✓
$v_6$	$v_1, v_5$	$w_6, w_1, w_2, w_7$	✓
$v_7$	$v_1, v_6$	$w_7, w_1, w_2, w_5$	✓

$$G = (V, E) \text{ is a tree with } 6 \text{ vertices}$$

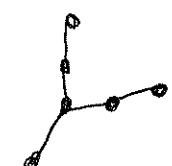
$$2|E| = \sum \deg(v_i)$$

$$2(6-1) = d_1 + d_2 + \dots + d_6$$

$$10 = d_1 + d_2 + \dots + d_6$$

$$|E| = 6-1 = 5$$

$$1 \leq d_i \leq 5$$



Q5. ( $5 \times 4$  pts.) a) If it exists, sketch a graph  $G_1$  which has 7 vertices  $\{v_1, v_2, \dots, v_7\}$  and 20 edges, and the degrees of the vertices are given by  $\deg(v_k) = k$  for  $k = 1, 2, \dots, 7$ .

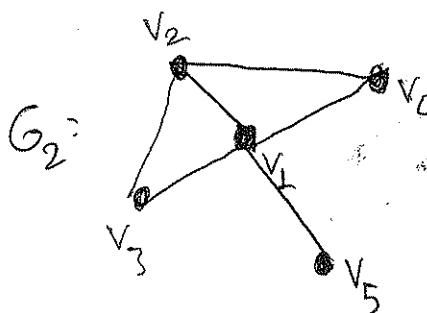
$G_1 = (V_1, E_1)$ ,  $|V_1| = 7$ ,  $|E_1| = 20 \Rightarrow$  By handshaking theorem  $2 \cdot |E_1| = \sum_{v \in V_1} \deg(v)$

$$\Rightarrow 2 \cdot 20 = \deg(v_1) + \deg(v_2) + \dots + \deg(v_7)$$

$$\Rightarrow 40 = 1 + 2 + 3 + \dots + 7 = \frac{7 \cdot (7+1)}{2} = 28 \Rightarrow 40 \neq 28 \text{ (contradiction)}$$

Thus, no such graph  $G_1$  exists.

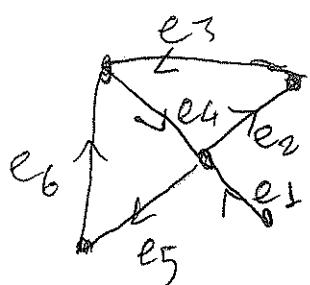
b) Sketch a simple graph (a graph with no loops and no multiple edges)  $G_2$  which has 5 vertices with degrees  $\deg(v_1) = 4$ ,  $\deg(v_2) = 3$ ,  $\deg(v_3) = \deg(v_4) = 2$ , and  $\deg(v_5) = 1$ .



c) Does the graph  $G_2$  in part (b) have an Euler circuit? If so, show the circuit by numbering its edges as  $e_1, e_2, \dots$  etc. ( $e_1$  is the first edge of the circuit,  $e_k$  is the  $k$ th edge.)

$\begin{cases} \deg(v_2) = 3 \\ \deg(v_5) = 1 \end{cases}$   $G_2$  has vertices with odd degrees, so  $G_2$  does not have an Euler circuit. (By the Theorem, A connected graph has an Euler circuit if and only if all vertices have even degrees).

d) Does the graph  $G_2$  in part (b) have an Euler path? If so, show the path by numbering its edges as  $e_1, e_2, \dots$  etc. ( $e_k$  is the  $k$ th edge of the path.)

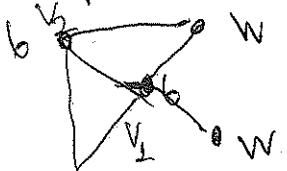


$e_1, e_2, \dots, e_6$  as shown is an Euler path.

e) Is the graph  $G_2$  in part (b) a bipartite graph?

If we try to color the vertices as black or white such that edges join vertices of opposite colors, we have (b: black, w: white)

Thus,  
it is not a bipartite  
graph.



But  $v_1$  and  $v_2$  are colored black and are joined by an edge.