

Department of Mathematics

Discrete Mathematics				
Midterm I				
Code	: Math 112		Last Name :	
Acad. Year	: 2017-2018		Name :	Student No :
Semester	: Spring		Department :	
Instructor	: Emelyanov, Finashin, Karayayla, Önal, Seven.		Signature :	
Date	: 29.03.2018		5 Questions on 4 Pages SHOW DETAILED WORK!	
Time	: 17.40			
Duration	: 110 minutes			
1	2	3	4	5

1. (8+12 pts) Suppose that a mathematics department offers 8 courses each semester (the same courses are offered each time), and these courses are distributed between 4 instructors so that each one teaches two courses every semester.

a) In how many ways can the courses be assigned to the instructors in one semester?

First instructor is assigned 2 courses in  $\binom{8}{2}$  ways, then 2nd instructor is assigned 2 of the remaining 6 courses in  $\binom{6}{2}$  ways, 3rd instructor in  $\binom{4}{2}$  ways and 4th instructor in  $\binom{2}{2}$  ways.

Answer:  $\binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = \frac{8!}{2! \cdot 6!} \cdot \frac{6!}{2! \cdot 4!} \cdot \frac{4!}{2! \cdot 2!} \cdot 1 = \frac{8!}{(2!)^4} = 7 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 21 \cdot 120 = \underline{\underline{2520}}$

b) In how many ways can the courses be assigned to the instructors in two semesters so that for each instructor the pair of courses he teaches in one semester is not the same as in the other semester (at least one course must be different)? We'll apply Inclusion Exclusion Principle:

After instructors are assigned 8 courses for the first semester in 2520 ways, let  $A_i =$  Set of distributions (assignments) in 2nd semester where  $i$ th instructor teaches the same 2 courses as in first semester.  $i=1, 2, 3, 4$ .

By Inclusion Exclusion; assignments for 2nd semester (after a choice of first semester) can be done as:

$N = 2520 - |A_1 \cup A_2 \cup A_3 \cup A_4| = 2520 - \sum_{i=1}^4 |A_i| + \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| - \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$

$|A_i| = \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = \frac{6!}{2! \cdot 2! \cdot 2!} = 90$  for all  $i$  (2 courses of instructor  $i$  are same, distribute the other 6 to 3 instr.)

$|A_i \cap A_j| = \binom{4}{2} \cdot \binom{2}{2} = 6$  for all  $i, j$

$|A_i \cap A_j \cap A_k| = \binom{2}{2} = 1$  (3 instructors teach the same 2 courses as in 1st sem, so 4th inst. also teaches the same 2 courses).

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$  (Everyone teaches the same courses).

Answer:  $2520 \cdot N = 2520 \cdot [2520 - 4 \cdot 90 + \binom{4}{2} \cdot 6 - \binom{4}{3} \cdot 1 + 1] = 2520 \cdot [2520 - 360 + 36 - 4 + 1] = 2520 \cdot 2193$

2. ( $2 \times 10$  pts) In how many ways can the 10 numerals  $0, 1, \dots, 9$  and the 26 letters  $A, \dots, Z$  be permuted (arranged in a row) in each of the following cases?

a) The numerals appear in the increasing order from left to right (there may be letters between numerals).

In the row of 36 characters, first choose the positions of the 10 numerals in  $\binom{36}{10}$  ways. Insert  $0, 1, 2, 3, \dots, 9$  in this order to the chosen 10 positions. (in 1 way!) Then arrange the 26 letters to remaining 26 positions in  $26!$  ways.

Answer:  $\binom{36}{10} \cdot 26!$

b) There is at least one letter between any pair of numerals.

This means no 2 numerals are adjacent.

First arrange the 26 letters in  $26!$  ways.

There are  $26+1=27$  spaces between the letters or on their right or left



choose 10 of these spaces in  $\binom{27}{10}$  ways and

insert the 10 numerals into those spaces in  $10!$  ways.

Answer:  $26! \cdot \binom{27}{10} \cdot 10!$

3. ( $2 \times 10$  pts) a) Find the coefficient of  $x^5$  in the expansion of  $(x^2 + x^{-1} - 2)^{10}$ .

Multinomial Expansion:  $(a+b+c)^{10} = \sum_{\substack{p+q+r=10 \\ p \geq 0, q \geq 0, r \geq 0}} \frac{10!}{p!q!r!} a^p b^q c^r \Rightarrow$  substitute  $a=x^2, b=x^{-1}, c=-2$ .

$x^5 = (x^2)^p \cdot (x^{-1})^q \Leftrightarrow 2p - q = 5$

$p \geq 0, q \geq 0, p+q = 10 - r \leq 10 \Rightarrow$

p	q	2p-q
3	1	5
4	3	5
5	5	5

→ These are all possible cases.

$(p, q) = (3, 1) \Rightarrow \frac{10!}{p!q!r!} (x^2)^p (x^{-1})^q (-2)^r = \frac{10!}{3! \cdot 1! \cdot 6!} x^5 \cdot (-2)^6$

$(p, q) = (4, 3) \Rightarrow \frac{10!}{p!q!r!} (x^2)^p (x^{-1})^q (-2)^r = \frac{10!}{4! \cdot 3! \cdot 3!} x^5 \cdot (-2)^3$

$(p, q) = (5, 5) \Rightarrow \frac{10!}{p!q!r!} (x^2)^p (x^{-1})^q (-2)^r = \frac{10!}{5! \cdot 5! \cdot 0!} x^5 \cdot (-2)^0 = \frac{10!}{5! \cdot 5!} x^5$

coefficient of  $x^5$  is:  $\frac{10!}{3! \cdot 6!} \cdot 2^6 + \frac{10!}{4! \cdot 3! \cdot 3!} \cdot 2^3 + \frac{10!}{5! \cdot 5!}$

3.b) Prove that for any integer  $0 \leq k \leq 20$  the following identity holds:

$$\begin{aligned}
 N &= \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{20}{k} = \binom{21}{k+1} \\
 &= \binom{k+1}{k+1} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{20}{k} \\
 &\stackrel{\text{Pascal's Identity}}{=} \binom{k+2}{k+1} + \binom{k+2}{k} + \dots + \binom{20}{k} \\
 &\stackrel{\text{Pascal's Identity}}{=} \binom{k+3}{k+1} + \binom{k+3}{k} + \dots + \binom{20}{k} \\
 &\vdots \\
 &= \binom{19}{k+1} + \binom{19}{k} + \binom{20}{k} = \binom{20}{k+1} + \binom{20}{k} = \binom{21}{k+1}
 \end{aligned}$$

At each step we use Pascal's identity  $\binom{a}{k+1} + \binom{a}{k} = \binom{a+1}{k+1}$

4. (2 x 10 pts) A collection of letters consists of A,A,B,B,C,C,D,E,F,G,H,I,J,K,L,M,N,O (totally 18 letters, containing 3 pairs of identical letters).

a) In how many ways can these letters be arranged around a circle? (Arrangements which differ by a circular rotation in the plane are considered to be the same.)



Any seating (arrangement) around a circle can be rotated so that letter D is in the position as shown (we have chosen D since it is not repeated). Taking this D as a reference point, the 17 remaining positions can be labelled from 1 to 17 (the positions become distinguishable now.)

We can arrange the other 17 letters involving 2 A's, 2 B's and 2 C's to these 17 distinguishable places in

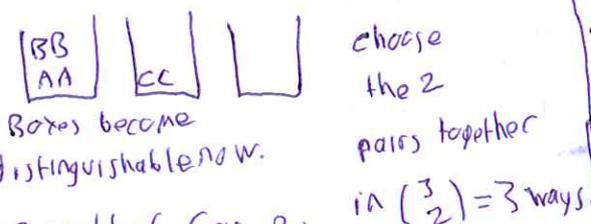
Answer:  $\frac{17!}{2! \cdot 2! \cdot 2!}$  ways (permutations with repetitions)

b) In how many ways can these 18 letters be split into 3 groups of equal size so that for each of the 3 pairs of identical letters, both of the identical letters appear in the same group (but for different pairs, the groups containing them may be different).

Case 1: Each pair in separate groups  $\boxed{AA} \quad \boxed{BB} \quad \boxed{CC}$  Here we distribute the letters to 3 non-distinguishable boxes (identical boxes). After A's, B's, C's are placed, boxes become distinguishable.

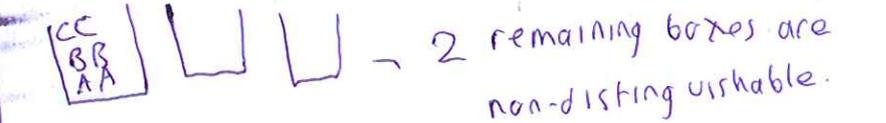
Result of case 1:  $\binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4} = \frac{12!}{4! \cdot 4! \cdot 4!}$  - Choose 4 to fill box with A's, choose 4 out of remaining 8 to fill box of B's, remaining 4 go to box of C's.

Case 2: 2 pairs in 1 box, 1 pair in another box



Result of Case 2:  $3 \cdot \binom{12}{2} \cdot \binom{10}{4} \cdot \binom{6}{6} = 3 \cdot \frac{12!}{2! \cdot 4! \cdot 6!}$

Case 3: All 3 pairs are in the same box (group)



12 distinct letters are separated into 2 groups of size 6 in  $\frac{1}{2} \cdot \binom{12}{6} \cdot \binom{6}{6} = \frac{1}{2} \cdot \frac{12!}{6! \cdot 6!}$

Answer:  $\frac{12!}{4! \cdot 4! \cdot 4!} + 3 \cdot \frac{12!}{2! \cdot 4! \cdot 6!} + \frac{1}{2} \cdot \frac{12!}{6! \cdot 6!}$

5. (6+7+7 pts) 20 identical diamonds are distributed to 8 people. In how many ways can this be done in each of the following cases?

a) Each person gets at least two diamonds.

Let person  $i$  take  $x_i$  diamonds.  $2 \leq x_i$  for all  $i$ ,  $x_1 + x_2 + \dots + x_8 = 20$

$$x_i = 2 + y_i \text{ for all } i = 1, 2, \dots, 8 \Rightarrow 0 \leq y_i \text{ for all } i \text{ and}$$

$$x_1 + x_2 + \dots + x_8 = 20$$

$$2 + y_1 + 2 + y_2 + \dots + 2 + y_8 = 20$$

$$16 + y_1 + y_2 + \dots + y_8 = 20$$

$$y_1 + y_2 + \dots + y_8 = 4, y_i \geq 0 \text{ for all } i = 1, 2, \dots, 8$$

Number of solutions is  $\binom{4+7}{4} = \binom{11}{4}$

Answer:  $\binom{11}{4}$

b) 6 people get an odd number of diamonds and 2 people get an even number of diamonds.

$$x_i = 2y_i + 1 \text{ for 6 values of } i \in \{1, 2, \dots, 8\} \text{ and}$$

$$x_i = 2y_i \text{ for 2 other values of } i \in \{1, 2, \dots, 8\}$$

$y_i \geq 0$  in any case.

$y_i \geq 0$  and  $y_i \in \mathbb{Z}$  in all cases. Choose 6 people to get odd number of diam.

$$x_1 + x_2 + \dots + x_8 = 20$$

$$\binom{8}{6} = \frac{8 \cdot 7}{2!} = 28 \text{ ways.}$$

$$2y_1 + 2y_2 + \dots + 2y_8 + 1 + 1 + 1 + 1 + 1 + 1 = 20$$

$$2(y_1 + y_2 + \dots + y_8) = 14$$

$$y_1 + y_2 + \dots + y_8 = 7, y_i \geq 0, y_i \in \mathbb{Z}$$

Number of solutions is  $\binom{7+8-1}{7} = \binom{14}{7}$

Answer:  $\binom{14}{7} \cdot 28$

c) Nobody gets exactly 6 diamonds.

$\left| \text{All cases} \right| - \left| \text{cases where at least one person gets exactly 6 diamonds} \right|$

$$= \binom{20+8-1}{20} - \left| A_1 \cup A_2 \cup \dots \cup A_8 \right| \text{ where } A_i = \{ \text{distributions where person } i \text{ gets exactly 6 diamonds} \}$$

$$= \binom{27}{20} - \left[ \sum_{i=1}^8 |A_i| - \sum_{1 \leq i < j \leq 8} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 8} |A_i \cap A_j \cap A_k| \right]$$

We do not have intersection of more than 3 of these sets since at most 3 people can get exactly 6 (There are 20 diam in total)

$$|A_i \cap A_j| = \binom{20-12+6-1}{20-12} = \binom{13}{8} \text{ - Give 6 to } i^{\text{th}} \text{ and } j^{\text{th}} \text{ person, distribute 8 to 6 people.}$$

$$|A_i \cap A_j \cap A_k| = \binom{20-18+5-1}{20-18} = \binom{6}{2} \text{ - Give 6 to } i^{\text{th}}, j^{\text{th}} \text{ and } k^{\text{th}} \text{ person}$$

$$|A_i| = \binom{20-6+7-1}{20-6} = \binom{20}{14} \text{ Distribute remaining 2 to 5 people.}$$

$$\text{Answer: } \binom{27}{20} - \binom{8}{1} \binom{20}{14} + \binom{8}{2} \binom{13}{8} - \binom{8}{3} \binom{6}{2}$$