M E T U
Department of Mathematics

	Complex Calculus Final	

Code : *Math 353* Acad. Year : *2017-2018* Semester : *Fall* Date : *January.13.2018* Time : *13:30* Duration : *135 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6		SHOW YOUR WORK	

Question 1 (9+4+5 pts) For the function

$$f(z) = z^5 \cos\left(\frac{4}{z^2}\right) + \frac{z^2+1}{(z-1)^2(z^2+9)},$$

a) Find all poles of $f(z)$. What are the orders of these poles, and what are the residues of $f(z)$ at these poles?

$f(z)$ is analytic everywhere except $z=0$, $z=1$, $z=3i$, $z=-3i$. Since $z^5\cos\frac{4}{z^2}$ is analytic at $z=1$ and $\frac{z^2+1}{z^2+9}$ is analytic at $z=1$ with $\frac{1^2+1}{1^2+9} = \frac{1}{5} \neq 0$ f has a pole of order 2 at $z=0$ with residue

$$\operatorname*{Res}_{z=1} f(z) = \operatorname*{Res}_{z=\cancel{0}\,1} \frac{z^2+1}{(z-1)^2(z^2+9)} = \left(\frac{z^2+1}{z^2+9}\right)'\Big|_{z=1} = \frac{2z(z^2+9)-(z^2+1)\cdot 2z}{(z^2+9)^2}\Big|_{z=1} = \frac{20-4}{100} = \frac{16}{100}$$

Since $z^5\cos\frac{4}{z^2}$ is analytic at $z=3i$ and $\frac{z^2+1}{(z-1)^2(z+3i)}$ is analytic at $z=3i$ with $\frac{(3i)^2+1}{(3i-1)^2\,6i} \neq 0$ f has a pole of order 1 (simple pole) at $z=3i$ with $\operatorname*{Res}_{z=3i} f(z) = \operatorname*{Res}_{z=3i} \frac{z^2+1}{(z-1)^2(z^2+9)} = \left(\frac{z^2+1}{(z-1)^2(z+3i)}\right)\Big|_{z=3i} = \frac{-8}{(-8-6i)6i} = \frac{2}{12i-9}$

Finally $z^5\cos\frac{4}{z^2}$ is analytic at $z=-3i$ and $\frac{z^2+1}{(z-1)^2(z-3i)}$ is analytic at $z=3i$ with $\frac{(-3i)^2+1}{(-3i-1)^2(-6i)} \neq 0$ so f has a simple pole at $z=-3i$ with $\operatorname*{Res}_{z=-3i} f(z) = \left(\frac{z^2+1}{(z-1)^2(z-3i)}\right)\Big|_{z=-3i} = \frac{-8}{(-8+6i)(-6i)} = \frac{-2}{12i+9}$

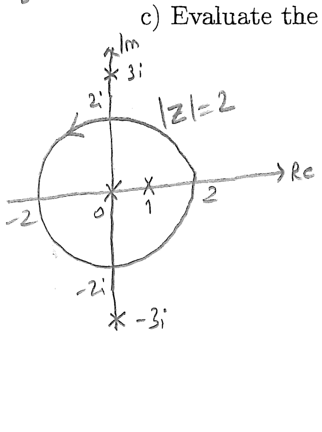
b) Does $f(z)$ have other singular points except for its poles? If so, what type of singular points are they, and what are the residues of $f(z)$ at these point(s)?

at $z=0$ $\frac{z^2+1}{(z-1)^2(z^2+9)}$ is analytic and hence doesnot contribute to the principal part of Laurent series of f at $z=0$

$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ $z \in \mathbb{C}$ hence $z^5\cos\frac{4}{z^2} = z^5 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{4^{2n}}{z^{4n}} = \sum_{n=0}^{\infty} \frac{(-16)^n}{(2n)!} z^{-4n+5}$ $|z|>0$ principal part of this series is equal to principal part of Laurent series of f at $z=0$ and principal part of the above series contains infinitely many negative ~~terms~~ powers of z so f has an essentol singularity at $z=0$. moreover the coefficient of $\frac{1}{z}$ is 0 so

$\operatorname*{Res}_{z=0} f(z) = 0$

c) Evaluate the contour integral $\int_C f(z)dz$ where C is the circle $|z| = 2$ oriented counter-clockwise.



$$\int_C f(z)dz = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right) = 2\pi i\left(0 + \frac{16}{100}\right) = \frac{32\pi i}{100}$$

Question 2 (10+8 pts) Let $f(z)$ be the linear fractional transformation such that $f(1) = 2i$, $f(2i) = -i$, $f(-i) = 1$.

OR use Cross Ratio

(a) Find a formula of $f(z)$.

$$\begin{array}{ccccc} 1 & \xrightarrow{h} & 1 & \xrightarrow{g} & 2i \\ 2i & \longrightarrow & 0 & \longrightarrow & -i \\ -i & \longrightarrow & \infty & \longrightarrow & 1 \end{array}$$

Let us find h^{-1} and g then $f = g \circ h$

$h^{-1}(z) = \frac{az+b}{cz+d}$ $\quad h^{-1}(\infty) = -i \Rightarrow a = -ci$

$h^{-1}(0) = 2i \Rightarrow b = 2di$

$h^{-1}(1) = \frac{-ci+2di}{c+d} = 1 \Rightarrow d(2i-1) = c(1+i) \Rightarrow$ so take $d = 1+i$ $c = 2i-1$

$h^{-1}(z) = \frac{(2+i)z + (-2+2i)}{(2i-1)z + (1+i)}$ so $h(z) = \frac{(1+i)z + (2-2i)}{(-2i+1)z + (2+i)}$

$g(z) = \frac{az+b}{cz+d}$

$g(0) = -i \Rightarrow b = -di$

$g(\infty) = 1 \Rightarrow a = c$

$g(1) = \frac{a-di}{a+d} = 2i \Rightarrow a(1-2i) = 3id \Rightarrow a = 3i$, $d = 1-2i$

So $g(z) = \frac{3iz + (-2-i)}{3iz + (1-2i)}$

then $(g \circ h)(z) = \dfrac{\frac{(-3+3i)z + (6+6i)}{(-2i+1)z + (2+i)} + (-2-i)}{\frac{(-3+3i)z + (6+6i)}{(-2i+1)z + (2+i)} + (1-2i)} = \frac{(6i-7)z + (3+2i)}{(i-6)z + (10+3i)}$

(b) Show that $f \circ f \circ f(z) = z$ for all z in the domain of f.

$(f \circ f \circ f)(1) = (f \circ f)(2i) = f(-i) = 1$

$(f \circ f \circ f)(2i) = (f \circ f)(-i) = f(1) = 2i$

$(f \circ f \circ f)(-i) = (f \circ f)(1) = f(2i) = -i$

So ~~f~~ $f \circ f \circ f$ fixes 3 points, but note that a Möbius Transformation is completely determined by its action on 3 points. Since ~~f(z)~~ $= z$ for $z = 1, 2i, -i$ $f(z) = z$ ~~for~~ all z in the domain of f. $f \circ f \circ f(z) = z$

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Question 3 (9+9 pts) a) Write down $f(z) = \dfrac{\exp(1/z) - 1}{z - 3}$ as a product of two Laurent series on the annulus $3 < |z| < \infty$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

$$e^{1/z} - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} z^{-n} \qquad |z| > 0$$

$$0 < \left|\frac{3}{z}\right| < 1$$

$$\frac{1}{z-3} = \frac{1/z}{1 - 3/z} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n = \sum_{n=0}^{\infty} 3^n z^{-n-1} \qquad 3 < |z| < \infty$$

Then $f(z) = \dfrac{e^{1/z} - 1}{z-3} = \left(\sum_{n=1}^{\infty} \frac{1}{n!} z^{-n}\right)\left(\sum_{n=0}^{\infty} 3^n z^{-n-1}\right)$ with region of convergence $\{z \mid |z| > 0\} \cap \{z \mid 3 < |z| < \infty\} = \{z \mid 3 < |z| < \infty\}$

(Note that g has a removable singularity at $z = 0$, hence defining $g(0) = 0$ makes $g(z)$ an entire function)

b) Let $g(z) = \dfrac{\sin(z^2)}{z}$ and $h(z) = zg'(z)$. Using Taylor series, calculate $h^{(353)}(0)$.

$g(z)$ has singularity at $z = 0$. Let us find its Laurent series at $z = 0$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \qquad z \in \mathbb{C}$$

$$g(z) = \frac{\sin z^2}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+1} \qquad |z| > 0$$

A power series, equality holds also at $z = 0$, hence for all $z \in \mathbb{C}$, function is entire

Differentiating w.r.t z we get $g'(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n+1)}{(2n+1)!} z^{4n}$ $\quad |z| > 0$

and multiplying by z gives $h(z) = z g'(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n+1)}{(2n+1)!} z^{4n+1}$

Observe that $zg'(z)$ does not contain any negative power term in its Laurent Series expansion so the series given is indeed Taylor series of $h(z)$ at $z = 0$ which is known to be equal to $\sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n$, then by uniqueness of Taylor Series two series must be equal implies termwise equality

$$\frac{h^{(353)}(0)\, z^{353}}{353!} = \frac{(-1)^{88}\, 353\, z^{353}}{177!}$$

Thus $h^{(353)}(0) = \dfrac{353\,(353!)}{177!}$

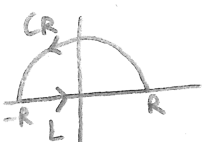
$(4n+1 = 353 \Rightarrow n = 88)$

Question 4 (10 pts) Suppose that $f(z)$ is an analytic function on the open unit disk D such that for any $z \in D - \{0\}$, $|f(z)| \le |\sin(1/z)|$ holds. Show that f must be the constant 0 function. (Hint: Can you find any $z \in D$ such that $f(z) = 0$?)

Observe that for $z_n = \frac{1}{2\pi n}$ $n \in \mathbb{Z}$ $\sin \frac{1}{z} = \sin(2\pi n) = 0$ so that $f(z_n) = 0$ because of the inequality. But recall that a nonconstant analytic function's zeros cannot have accumulation points, yet 0 is an accumulation point of $\{\frac{1}{2\pi n} \mid n \in \mathbb{Z}\}$ thus f must be constant 0 function on D.

Question 5 (18 pts) Use residues to find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx$. (Hint: Integrate a suitably chosen function on a suitable closed contour and use Jordan's Lemma). Do not Simplify Residue(s) you find and the final answer!

Let $f(z) = \frac{(z+1)e^{iz}}{z^2+4z+5}$ Note that f has two poles one at $z = -2+i$ the other at $z = -2-i$



and $\frac{(z+1)e^{iz}}{(z+2+i)}\Big|_{z=-2+i} = \frac{(-1+i)e^{-2i-1}}{2i} \neq 0$ so f has a simple pole at $z=-2+i$

with $\text{Res}_{z=-2+i} f(z) = \left(\frac{(z+1)e^{iz}}{z+2+i}\right)\Big|_{z=-2+i} = \frac{(i-1)}{2i} e^{-2i-1}$ ✓ $\frac{i+1}{2} e^{-2i-1}$

Integrating along the contour given above

$$\int_{C_R} f(z)dz + \int_L f(z)dz = 2\pi i \operatorname*{Res}_{z=-2+i} f(z) \quad (*)$$

parametrize L as $\gamma(t) = t$ $-R < t < R$

then $\int_L f(z)dz = \int_{-R}^{R} \frac{(t+1)e^{it}}{t^2+4t+5}dt$ which has the real part

$\int_{-R}^{R} \frac{(t+1)\cos t}{t^2+4t+5}dt$. moreover for $z \in C_R$

$$\left|\frac{z+1}{z^2+4z+5}\right| \le \frac{|z|+1}{|z+2|^2-1} \le \frac{R+1}{(|z|-2)^2-1} = \frac{R+1}{R^2-4R+3} = M_R$$

Observe that $\lim_{R\to\infty} M_R = \lim_{R\to\infty} \frac{1+\frac{1}{R}}{R-4+\frac{3}{R}} = 0$ then by Jordan's Lemma

$\lim_{R\to\infty}\left|\int_{C_R} \frac{(z+1)e^{iz}}{z^2+4z+5}\right| = 0$. Moreover $\left|\text{Re}\int_{C_R} \frac{(z+1)e^{iz}}{z^2+4z+5}dz\right| \le \left|\int_{C_R}\frac{(z+1)e^{iz}}{z^2+4z+5}\right|$

So $\lim_{R\to\infty} \text{Re}\int_{C_R} \frac{(z+1)e^{iz}}{z^2+4z+5} = 0$

Now take real part of (*)

$$\text{Re}\int_{C_R} f(z)dz + \int_{-R}^{R} \frac{(t+1)\cos t}{t^2+4t+5}dt = \text{Re}\left\{\frac{i+1}{2}e^{-2i-1}\right\}$$

Now, take $R \to \infty$ first summand will have limit 0 and we get

$$\text{P.V} \int_{-\infty}^{\infty} \frac{(t+1)\cos t}{t^2+4t+5}dt = \text{Re}\left\{\frac{i+1}{2}e^{-2i-1}\right\}$$

Question 6 (5+13 pts) a) Show that 0 is a simple pole of $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$ where $(a \neq b)$ and calculate the residue of $f(z)$ at 0.

$$e^{iaz} = \sum_{n=0}^{\infty} \frac{(iaz)^n}{n!}, \; z \in \mathbb{C} \text{ then } \frac{e^{iaz} - e^{ibz}}{z^2} = \sum_{n=0}^{\infty} \frac{(ia)^n - (ib)^n}{n!} z^{n-2} = \sum_{n=1}^{\infty} \frac{(ia)^n - (ib)^n}{n!} z^{n-2} = \sum_{n=0}^{\infty} \frac{(ia)^{n+1} - (ib)^{n+1}}{(n+1)!} z^{n-1}$$

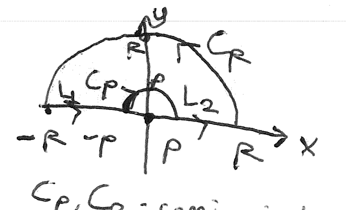
0th term has coeff. 0

So the principal part of Laurent series of f at $z=0$ is $\dfrac{(ia-ib)}{z}$ so f has a simple pole at $z=0$ with $\operatorname{Res}_{z=0} f(z) = (ia-ib)$

b) Derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \quad (a \geq 0, b \geq 0)$$

by considering the contour integral of $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$ on the indented path $C_R + L_1 + C_\rho + L_2$ as shown in the figure. You will need to use the residue(s) of $f(z)$ in your calculations.



C_ρ, C_R: semi-circles.

for $z \in C_R$

$$\left| \frac{e^{iaz} - e^{ibz}}{z^2} \right| \leq \frac{|e^{iaz}| + |e^{ibz}|}{R^2} = \frac{e^{-a \operatorname{Im} z} + e^{-b \operatorname{Im} z}}{R^2} \leq \frac{2}{R^2}$$

since $\operatorname{Im} z \geq 0$ for $z \in C_R$

and $0 \leq \left| \int_{C_R} f(z) dz \right| \leq \max_{z \in C_R} |f(z)| \cdot \pi R \leq \dfrac{2\pi R}{R^2} = M_R$, $\lim_{R \to \infty} M_R = 0$ so $\lim_{R \to \infty} \int_{C_R} f(z) dz = 0$ (by Squeeze Thm)

for $z \in C_\rho$ since $z=0$ is a simple pole of $f(z)$ we have

$$\lim_{\rho \to 0} \int_{C_\rho} f(z) dz = -\pi i \operatorname{Res}_{z=0} f(z) = \pi(a-b)$$

Parametrize $L_2: \gamma(t) = t \quad \rho \leq t \leq R$

$-L_1: \gamma(t) = -t \quad \rho \leq t \leq R$

hence

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_\rho^R \frac{e^{iat} - e^{ibt}}{t^2} dt + \int_\rho^R \frac{e^{-iat} - e^{-ibt}}{t^2} dt = \int_\rho^R \frac{2\cos at - 2\cos bt}{t^2} dt$$

Now let us use all the information to integrate on the path $C_R + L_1 + C_\rho + L_2$

$$\int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 0$$

since $f(z)$ has only singularity at $z=0$ outside the contour (Cauchy Goursat Thm)

Take limit as $R \to \infty, \rho \to 0$ then first term disappears, second term converges to $\pi(a-b)$ and we get

$$\int_0^{\infty} \frac{2\cos at - 2\cos bt}{t^2} dt = \pi(b-a) \quad \text{so} \quad \int_0^{\infty} \frac{\cos at - \cos bt}{t^2} dt = \frac{\pi(b-a)}{2}$$