M E T U
Department of Mathematics

	Complex Calculus **Final**	

Code : *Math 353* Acad. Year : *2017-2018* Semester : *Fall* Date : *January.13.2018* Time : *13:30* Duration : *135 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	**SHOW YOUR WORK**	

Question 1 (9+4+5 pts) For the function

$$f(z) = z^5 \cos\left(\frac{4}{z^2}\right) + \frac{z^2+1}{(z-1)^2(z^2+9)},$$

a) Find all poles of $f(z)$. What are the orders of these poles, and what are the residues of $f(z)$ at these poles?

b) Does $f(z)$ have other singular points except for its poles? If so, what type of singular points are they, and what are the residues of $f(z)$ at these point(s)?

c) Evaluate the contour integral $\int_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is the circle $|z| = 2$ oriented counter-clockwise.

Question 2 (10+8 pts) Let $f(z)$ be the linear fractional transformation such that $f(1) = 2i$, $f(2i) = -i$, $f(-i) = 1$.

(a) Find a formula of $f(z)$.

(b) Show that $f \circ f \circ f(z) = z$ for all z in the domain of f.

Question 3 (9+9 pts) a) Write down $f(z) = \dfrac{\exp(1/z) - 1}{z - 3}$ as a product of two Laurent series on the annulus $3 < |z| < \infty$.

b) Let $g(z) = \dfrac{\sin(z^2)}{z}$ and $h(z) = zg'(z)$. Using Taylor series, calculate $h^{(353)}(0)$.

Question 4 (10 pts) Suppose that $f(z)$ is an analytic function on the open unit disk D such that for any $z \in D - \{0\}$, $|f(z)| \leq |\sin(1/z)|$ holds. Show that f must be the constant 0 function. (Hint: Can you find any $z \in D$ such that $f(z) = 0$?)

Question 5 (18 pts) Use residues to find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5} dx$. (Hint: Integrate a suitably chosen function on a suitable closed contour and use Jordan's Lemma).

Question 6 (5+13 pts) a) Show that 0 is a simple pole of $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$ where $(a \neq b)$ and calculate the residue of $f(z)$ at 0.

b) Derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b - a) \quad (a \geq 0, b \geq 0)$$

by considering the contour integral of $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$ on the indented path $\mathcal{C}_R + L_1 + \mathcal{C}_\rho + L_2$ as shown in the figure. You will need to use the residue(s) of $f(z)$ in your calculations.