Math 115 Analytic Geometry

Exercises

(Vectors)

1. Calculate the direction cosines of the vector $\vec{a} = (3, 12, 4)$.

2. Is it possible for a vector to make the following angles with the coordinate axes?
   a) $\alpha = 45°, \beta = 60°, \gamma = 120°$,
   b) $\alpha = 45°, \beta = 135°, \gamma = 60°$,
   c) $\alpha = 90°, \beta = 150°, \gamma = 60°$,
   d) $\alpha = 30°, \beta = 45°, \gamma = 45°$,
   e) $\alpha = 45°, \beta = 45°, \gamma = 45°$,
   f) $\alpha = 30°, \beta = 45°$,
   g) $\beta = 60°, \gamma = 60°$,
   h) $\alpha = 150°, \gamma = 30°$,
   where α, β and γ are the angles between the vector and $x-$axis, $y-$axis and $z-$axis respectively.

3. A vector makes angles $\alpha = 120°$ and $\gamma = 45°$ with the $x-$axis and $z-$axis respectively. What angle does it make with the $y-$axis?

4. Determine the components of the vector which makes equal angles with the coordinate axes and whoso modulus (length) is 2.

5. Given that $|\vec{a}| = 13$, $|\vec{b}| = 19$ and $|\vec{a} + \vec{b}| = 24$, find $|\vec{a} - \vec{b}|$.

6. Given that $|\vec{a}| = 11$, $|\vec{b}| = 23$ and $|\vec{a} - \vec{b}| = 30$, find $|\vec{a} + \vec{b}|$.

7. Vectors $\vec{a}$ and $\vec{b}$ are perpendicular. If $|\vec{a}| = 5$ and $|\vec{b}| = 12$, compute $|\vec{a}+\vec{b}|$ and $|\vec{a}-\vec{b}|$.

8. Vectors $\vec{a}$ and $\vec{b}$ make an angle of measure $60°$. If $|\vec{a}| = 3$ and $|\vec{b}| = 5$, compute $|\vec{a}+\vec{b}|$ and $|\vec{a} - \vec{b}|$.

9. Vectors $\vec{a}$ and $\vec{b}$ make an angle of measure $120°$. If $|\vec{a}| = 3$ and $|\vec{b}| = 5$, compute $|\vec{a} + \vec{b}|$ and $|\vec{a} - \vec{b}|$.

10. What condition must be satisfied by $\vec{a}$ and $\vec{b}$ so that $\vec{a} + \vec{b}$ bisects the angle between $\vec{a}$ and $\vec{b}$?

11. What condition must be satisfied by $\vec{a}$ and $\vec{b}$ so that
    a) $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$,
    b) $|\vec{a} + \vec{b}| > |\vec{a} - \vec{b}|$,
    c) $|\vec{a} + \vec{b}| < |\vec{a} - \vec{b}|$?

12. Let G be the centroid of the triangle ABC. Show that $\vec{GA} + \vec{GB} + \vec{GC} = 0$.

13. Determine the values of λ and μ for which the vectors $\vec{a} = -2\vec{i} + 3\vec{j} + \lambda\vec{k}$ and $\vec{b} = \mu\vec{i} + 6\vec{j} + 2\vec{k}$ are parallel?

14. Determine the components of the vector $\vec{c}$ which bisects the angle between the vectors $\vec{a} = (2, -3, 6)$ and $\vec{b} = (-1, 2, -2)$ if $|\vec{c}| = 3\sqrt{42}$.

15. If $\vec{u}$ and $\vec{v}$ are non-parallel vectors, then every vector $\vec{a}$ in the plane of $\vec{u}$ and $\vec{v}$ can be expressed in the form $\vec{a} = \alpha\vec{u} + \beta\vec{v}$. Prove that α and β are determined uniquely by $\vec{u}$ and $\vec{v}$. (The representation of $\vec{a}$ in the form $\vec{a} = \alpha\vec{u} + \beta\vec{v}$ is called the resolution of $\vec{a}$ into components with respect to the basis $\{\vec{u}, \vec{v}\}$; the real numbers α and β are called the coefficients of this resolution.)

16. Given the vectors $\vec{u} = (2, -3)$ and $\vec{v} = (1, 2)$ in the plane. Find the resolution of the vector $\vec{a} = (9, 4)$ with respect to the basis $\{\vec{u}, \vec{v}\}$.

17. Given the vectors $\vec{a} = (3, -1)$, $\vec{b} = (1, -2)$ and $\vec{c} = (-1, 7)$ in the plane. Determine the resolution of the vector $\vec{u} = \vec{a} + \vec{b} + \vec{c}$ with respect to the basis $\{\vec{a}, \vec{b}\}$.

18. Given the vectors $\vec{u} = (3, -2, 1)$, $\vec{v} = (-1, 1, 2)$ and $\vec{w} = (2, 1, -3)$. Determine the resolution of the vector $\vec{a} = (11, -6, 5)$ with respect to the basis $\{\vec{u}, \vec{v}, \vec{w}\}$.

19. Vectors $\vec{a}$ and $\vec{b}$ are perpendicular and each of them makes an angle of measure $\pi/3$ with a certain vector $\vec{c}$. If $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 8$, calculate
 a) $(3\vec{a} - 2\vec{b})(\vec{b} + 3\vec{c})$,
 b) $(\vec{a} + \vec{b} + \vec{c})^2$,
 c) $(\vec{a} + 2\vec{b} - 3\vec{c})^2$.

20. For any two vectors $\vec{a}$ and $\vec{b}$, prove that

$$(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = 2(\vec{a}^2 + \vec{b}^2).$$

 Find the geometric interpretation of this equality.

21. Given the unit vectors $\vec{a}$ $\vec{b}$ and $\vec{c}$ which satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. Compute $\vec{a}\vec{b} + \vec{b}\vec{c} + \vec{c}\vec{a}$.

22. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 1$ and $|\vec{c}| = 4$, evaluate $\vec{a}\vec{b} + \vec{b}\vec{c} + \vec{c}\vec{a}$.

23. Measure of the angle between any two of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$is $\pi/3$ radians. If $|\vec{a}| = 4$, $|\vec{b}| = 2$ and $|\vec{c}| = 6$, evaluate $|\vec{a} + \vec{b} + \vec{c}|$.

24. Given that $|\vec{a}| = 3$ and $|\vec{b}| = 5$. Determine the value of α so that the vectors $\vec{a} + \alpha\vec{b}$ and $\vec{a} - \alpha\vec{b}$ are perpendicular.

25. A vector $\vec{u}$ is perpendicular to the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = 18\vec{i} - 22\vec{j} - 5\vec{k}$ and makes an obtuse angle with the y-axis. Find the components of $\vec{u}$ if $|\vec{u}| = 14$.

26. Given the vectors $\vec{a} = (2, -1, 3)$, $\vec{b} = (1, -3, 2)$ and $\vec{c} = (3, 2, -4)$. Find the vector $\vec{x}$ satisfying $\vec{x} \cdot \vec{a} = -5$, $\vec{x} \cdot \vec{b} = -11$, $\vec{x} \cdot \vec{c} = 20$.

27. Given the vectors $\vec{a} = (1, -3, 4)$, $\vec{b} = (3, -4, 2)$ and $\vec{c} = (-1, 1, 4)$. Find the projection of $\vec{a} + \vec{b}$ on $\vec{c}$.

28. If $\vec{a}$ and $\vec{b}$ are perpendicular and $|\vec{a}| = 3$, $|\vec{b}| = 5$, find
a) $|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$,
b) $|(3\vec{a} - \vec{b}) \times ((\vec{a} - 2\vec{b})|$.

29. What condition must be satisfied by $\vec{a}$ and $\vec{b}$ so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are collinear (parallel)?

30. For any two vectors $\vec{a}$ and $\vec{b}$, prove that
a) $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$,
b) $|\vec{a} \times \vec{b}|^2 \leq |\vec{a}|^2 |\vec{b}|^2$.

31. If $\vec{a} + \vec{b} + \vec{c} = 0$, prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

32. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, prove that $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ are parallel.

33. Given the points $A(1, 2, 0)$, $B(3, 0, -3)$ and $C(5, 2, 6)$. Compute the area of the triangle ABC.

34. Given the vectors $\vec{u} = (2, -3, 1)$, $\vec{v} = (-3, 1, 2)$ and $\vec{w} = (1, 2, 3)$. Find $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$.

35. For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, prove that $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$.

36. For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, prove that $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

37. What conditions must be satisfied by $\vec{u}$, $\vec{v}$ and $\vec{w}$ so that $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$?

38. For $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{z}$, prove the following identities
a)$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = 0$,
b)$(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{z}) = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{z}) - (\vec{u} \cdot \vec{z})(\vec{v} \cdot \vec{w})$,
c)$(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{z}) + (\vec{u} \times \vec{w}) \cdot (\vec{v} \times \vec{z}) + (\vec{u} \times \vec{z}) \cdot (v \times \vec{w}) = 0$,
d)$\vec{u} \times (\vec{v} \times (\vec{w} \times \vec{z})) = (\vec{v} \cdot \vec{z})(\vec{u} \times \vec{w}) - (\vec{v} \cdot \vec{w})(\vec{u} \times \vec{z})$.

39. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ show that

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

and define $[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

40. Show that the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are right handed if and only if $[\vec{a}, \vec{b}, \vec{c}] > 0$.

41. Prove that for any $\vec{a}$, $\vec{b}$, $\vec{c}$,

$$[(\vec{a}+\vec{b}),(\vec{b}+\vec{c}),(\vec{c}+\vec{a})] = 2[\vec{a},\vec{b},\vec{c}].$$

42. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$, prove that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

43. Show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c}]=0$.

44. Prove that the points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ lie in the same plane.

Answers

3) $60°$ 4) $(\frac{2\sqrt{3}}{3},\frac{2\sqrt{3}}{3},\frac{2\sqrt{3}}{3})$ 5) 22 6) 20 7) $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|=13$
8) $|\vec{a}+\vec{b}|=7$ and $|\vec{a}-\vec{b}|=\sqrt{19}$ 9) $|\vec{a}+\vec{b}|=\sqrt{19}$ and $|\vec{a}-\vec{b}|=7$ 10) $|\vec{a}|=|\vec{b}|$
11) a) vectors are perpendicular, b) the angle between the vectors is acute, c) the angle between the vectors is obtuse. 12) Use the fact that $3\vec{G}=\vec{A}+\vec{B}+\vec{C}$ 13) $\lambda=1$, $\mu=-4$
14) $(-3,15,12)$ 16) $\vec{a}=2\vec{u}+5\vec{v}$ 17) $\vec{u}=2\vec{a}-3\vec{b}$ 18) $\vec{a}=\frac{19}{5}\vec{u}+\frac{6}{5}\vec{v}+\frac{2}{5}\vec{w}$
19) a) $-79/2$ b) 177 c) 403 21) $-3/2$ 22) -13 23) 10 24) $3/5$ 25) $(-4,-6,12)$
26) $(2,3,-2)$ 27) $(-13/18,13/18,26/9)$ 28) a) 30 b) 75 29) $|\vec{a}|=|\vec{b}|$
33) 14 34) $(-7,14,-7)$, $(10,13,19)$

Math 115

1.

(a) Write an equation for the line ℓ through the points $(-1,-2)$ and $(11,-6)$.
(b) Write an equation for the line m through the point $(13,0)$ perpendicular to ℓ.
(c) If n has slope 3 and y-intercept 1 find an equation for the line n.
(d) If $A(-1,-2)$, $B(11,-6)$ and $C(13,0)$ are three consecutive vertices of a parallelogram $ABCD$ find the length of the diagonal $\overline{BD}$ without finding D. (Explain)

2. Sketch the graphs of the given relations in the xy-coordinate plane.
(a) $R = \{(x,y) \mid y \geq x^2 \text{ and } y \leq x+2\}$.
(b) $S = \{(r,\theta) \mid 0 \leq \theta \leq \frac{\pi}{2},\ r=-2\}$.
(c) $T = \{(r,\theta) \mid 0 \leq r \leq 2\sin\theta,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}\}$.

3. Plot the following points : $(-3,-\pi/6)$, $(2,7\pi/4)$, $(-1,\pi/3)$ in the polar plane.

4. Let P be a point in the plane with a pair of polar coordinates $(-2,3\pi/4)$.
(a) Find the Cartesian coordinates of P.
(b) Find a polar equation for the line ℓ through $(-2,3\pi/4)$ perpendicular to the line $\theta=\pi/2$, i.e. the $y-$axis.

5.

(a) Assume that $\bar{x}\bar{y}$- coordinate system is a translation of the xy-coordinate system. If the origin $\bar{O}$ of the $\bar{x}\bar{y}$- coordinate system has xy-coordinates $(-3,4)$ express xy-coordinates (x,y) of a point P in terms of $\bar{x}\bar{y}$- coordinates $(\bar{x},\bar{y})$ of P.
(b) Assume $\tilde{x}\tilde{y}$-coordinate system is obtained from the $\bar{x}\bar{y}$-coordinate system by a rotation through $\alpha=\pi/6$. Express $\bar{x}\bar{y}$-coordinates $(\bar{x},\bar{y})$ of a point P in terms of $\tilde{x}\tilde{y}$- coordinates $(\tilde{x},\tilde{y})$ of P.
(c) If $\ell = \{(x,y) \mid y=5\}$ then express C in terms of $\tilde{x}\tilde{y}$-coordinates.

6. Assume that $\bar{x}\bar{y}$- coordinate system is obtained from the xy-coordinate system by a rotation through $\alpha = tan^{-1}(3/4)$.
(a) Write $cos(\alpha)$ and $sin(\alpha)$.
(b) Express x and y in terms of $\bar{x}$ and $\bar{y}$.
(c) Let ℓ be the line $3x-4y+50=0$. What is the equation of ℓ in the $\bar{x}\bar{y}-$ coordinate system?
(d) If a point P has $xy-$ coordinates $(4,3)$ find its $\bar{x}\bar{y}-$coordinates.
(e) If possible, use (d) to find the distance from P to the $\bar{y}$-axis.