Analytic Geometry Exercises
Lines, Planes, Distances
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1. Find all points on the line $3x - 5y = 12$ which are at a distance of 3 units from the line $4x - 3y = 2$.

2. Find all points on the circle $x^2 + y^2 - 10x + 2y = 75$ whose distance from the line $x + y = 6$ is $2\sqrt{2}$.

3. What is the distance between the point $P(2,3,5)$ and the plane T given by $x - 2y + 4z = 23$? Find the point Q on T which is closest to P.

4. Find all points on the line $L : \vec{P} = (x, y, z) = (4 + t, 5 - t, 7 + 2t),\ t \in \mathbb{R}$ which are at a distance of 5 units from the plane $2x - 4y + 5z = \sqrt{5}$.

5. Find the distance from $P(3,4,7)$ to the line $L : \vec{P} = (x, y, z) = (1 + t, 1 + 2t, 3 + 3t)$, $t \in \mathbb{R}$, and find the point $Q \in L$ which is the closest point to P among all points on L.

6. Let two lines L_1 and L_2 be given by the symmetric equations

$$L_1 : x = \frac{3 - y}{2} = \frac{2z + 1}{4} \text{ and } L_2 : \frac{x + 3}{4} = \frac{z - 5}{8},\ y = 2.$$

a) What are direction vectors of L_1 and L_2?
b) Show that L_1 and L_2 are skew lines.
c) Write down the equations of the two planes T_1 and T_2 such that $L_1 \subset T_1$, $L_2 \subset T_2$ and $T_1 // T_2$.
d) Use equation of T_1 you found in part (c) and a chosen point $P_2 \in L_2$ to calculate the distance between L_1 and L_2.
e) Can we generalize the situation in part (c) to any pair of skew lines L_1 and L_2 in $\mathbb{R}^3$? In other words, if L_1 and L_2 are skew lines, can we find planes T_1 and T_2 such that $L_1 \subset T_1$, $L_2 \subset T_2$ and $T_1 // T_2$? Are these planes T_1 and T_2 unique?

7. Let two lines L_1 and L_2 be given by

$L_1 : \vec{r}_1(t) = (x, y, z) = (2+t, 3-t, 5+t), t \in \mathbb{R}$ and $L_2 : \vec{r}_2(t) = (x, y, z) = (1+2t, 1+3t, 1+4t), t \in \mathbb{R}$

a) Show that L_1 and L_2 are skew lines.
b) Find the distance between L_1 and L_2.
c) Find the points $P \in L_1$ and $Q \in L_2$ such that the distance between P and Q is the distance d between L_1 and L_2 (P and Q are the closest points to each other while one is on L_1 and the other is on L_2).

8. For two lines L_1 and L_2 given by $L_1 : \vec{P} = (x, y, z) = \vec{P}_1 + t\vec{v}_1,\ t \in \mathbb{R}$ and $L_2 : \vec{P} = (x, y, z) = \vec{P}_2 + t\vec{v}_2,\ t \in \mathbb{R}$, when L_1 and L_2 are skew lines, the distance d between L_1

and L_2 is given by the formula

$$d = \frac{|\vec{P_1P_2} \cdot \vec{n}|}{|\vec{n}|} = \frac{|\vec{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|}$$

where $\vec{n} = \vec{v}_1 \times \vec{v}_2$ (Indeed, $\vec{n}$ is a normal vector of the planes $T_1 // T_2$ such that $L_1 \subset T_1$ and $L_2 \subset T_2$ for the skew lines L_1 and L_2).

a) Why doesn't this formula work when the lines L_1 and L_2 are parallel or coincident?

b) Explain why the distance d between two skew lines is always positive. (What can you say about two lines L_1 and L_2 when the distance between them is 0?)

c) If $\vec{v}_1$ and $\vec{v}_2$ are not parallel vectors, then $\vec{n} = \vec{v}_1 \times \vec{v}_2 \neq \vec{0}$. In this case, if we find $d = 0$ from the above formula, what is the conclusion about the lines L_1 and L_2 (are they coincident, parallel, skew or intersecting)?

d) If $\vec{v}_1$ and $\vec{v}_2$ are not parallel vectors, in the formula given above $d = 0$ if and only if $\vec{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) = 0$. Combine this observation and your conclusion in part (c) to obtain the following criterion:

If two lines L_1 and L_2 are not parallel or coincident (that is, if their direction vectors $\vec{v}_1$ and $\vec{v}_2$ are not parallel vectors), then L_1 and L_2 are intersectiong lines if and only if $\vec{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) = 0$ where $P_1 \in L_1$ and $P_2 \in L_2$ are two points arbitrarily chosen from the lines L_1 and L_2.

(Note that $\vec{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) = 0$ means that the triple product of these three vectors is 0, and as a consequence, the vectors $\vec{P_1P_2}$, $\vec{v}_1$ and $\vec{v}_2$ are coplanar (they lie on the same plane).)

9. Find the distance between the lines $L_1 : \vec{P} = (x, y, z) = (3 + t, 2 - 5t, 3t)$, $t \in \mathbb{R}$ and $L_2 : x + 1 = \dfrac{1 - 3y}{15} = \dfrac{z + 8}{3}$.

10.a) Let L be a line in $\mathbb{R}^2$ and P_0 be a point in $\mathbb{R}^2$ such that $P_0 \notin L$. Characterize all lines $\tilde{L}$ in $\mathbb{R}^2$ such that $\tilde{L}$ passes through P_0 and $\tilde{L}$ intersets L. Is it true that the set of all such lines $\tilde{L}$ is the set of all lines through P_0?

b) Let L be a line in $\mathbb{R}^3$ and P_0 be a point in $\mathbb{R}^3$ such that $P_0 \notin L$. Characterie all lines $\tilde{L}$ such that $\tilde{L}$ passes through P_0 and $\tilde{L}$ intersects L. Is the set of all such lines $\tilde{L}$ equal to the set of lines on a certain plane which all pass through P_0, or do we need to exclude certain lines from this set? Which plane is this, and which line(s) should be excluded? What is the union of all such lines $\tilde{L}$ as a subset of $\mathbb{R}^3$?.

11. Let $A = (1, 0, 2)$ and L_1 be the line given by $L_1 : \vec{P} = (x, y, z) = (1 + 3t, 1 + 4t, 1 + 5t)$, $t \in \mathbb{R}$. For each line L_2 given below, find a line L (if such a line L exists at all) such that $A \in L$ and L intersects both of the lines L_1 and L_2.

a) $L_2 : \vec{P} = (x, y, z) = (5 + 2t, 5 + t, 3 + 2t)$, $t \in \mathbb{R}$. (There is a unique line L in this case.)

b) $L_2 : \vec{P} = (x, y, z) = (1 + 3t, -1 + 5t, 5 + 2t)$, $t \in \mathbb{R}$. (No such line L exists in this case.)

c) $L_2 : \vec{P} = (x, y, z) = (2, 3, 4) + t(4, 5, 7)$, $t \in \mathbb{R}$. (No such line L exists in this case).

12. When a line L in $\mathbb{R}^3$ is given (by a vector equation, or parametric equations for example), how can you describe it as the intersection of two planes? You can do it in nfinitely many different ways, but a very fast way of fnding two such planes is to look

at the symmetric equations of the line. Let $L: \dfrac{x+x_0}{v_1} = \dfrac{y-y_0}{v_2} = \dfrac{z-z_0}{v_3}$ be symmetric equations of L (Note that if one or two of v_1, v_2 or v_3 are 0, symmetric equations have other forms). From these equations, can you immediately write down equations of two planes which both contain L? What are they?
If one or two of the components of a direction vector $\vec{v} = (v_1, v_2, v_3)$ of L are zero, the form of symmetric equations of L is different. For example, if $v_2 = 0$, the symmetric equations of L are $L: \frac{x-x_0}{v_1} = \frac{z-z_0}{v_3}$, $y = y_0$. What are equations of two distinct planes each contining the line L?

Comments and hints for some of the problems

Problem 1. You should find 2 such points.
Problem 2. You should find 4 points in your answer.
Problem 4. You should find 2 points.
Problem 7. (part (c)) The quickest solution is by expressing P and Q in terms of parameters t (for P) and s (for Q) and then obtaining a system of 2 equations in 2 unknowns s and t using certain perpendicularity properties about P, Q and direction vectors of L_1 and L_2.
Problem 11. Note that one approach to solve this question is as follows: If such a line L exists, let $L \cap L_1 = B$ and $L \cap L_2 = C$. Using equations of L_1 and L_2, B can be written in terms of t and C can be written in terms of s where t and s are two parameters. Then, A, B, C are on the same line L iff (if and only if) $\vec{AB} \times \vec{AC} = \vec{0}$. This completely translates the problem to the solution of a system of 3 equations in two unknowns s and t. But the equations are not linear in s and t, they are of the form $ast + bs + ct + d = 0$. Such a system can be solved systematically, but a more geometric approach for the solution is as described:
Consider the plane T containing A and L_1. If L exists, then L must be on T (why?). If L_2 does not lie on T and $Q = L \cap L_2$, then $Q = L_2 \cap T$ (why?). Draw a picture illustrating L_1, L_2, A and T. Then try to solve the problem using these hints. You will need to write down the equation of T. You should also be careful when you find an intersection point C of T and L_2. Even if there is an intersection point C, it does not mean that L exists (see part (b) of the question). What do you need to check after finding C to conclude that L exists?