METU
Department of Mathematics

	Analytic Geometry Midterm II	
Code : *Math 115* Acad. Year : *2017-2018* Semester : *Fall* Coordinator: *E. Coskun* Date : *14.12.2017* Time : *17.40* Duration : *120 minutes*	Last Name : Name : Student No : Department : Signature : 5 Questions on 4 Pages Total **100** Points	

1	2	3	4	5		

1.(7+7+6 pts.) Let two lines L_1 and L_2 be given as follows

$$L_1 : \frac{x+2}{2} = \frac{y-1}{3} = \frac{z+1}{-1} \quad \text{and} \quad L_2 : \frac{x-1}{-1} = \frac{y+1}{2} = \frac{z-2}{4}.$$

a) Show that L_1 and L_2 are skew lines.

The direction vectors of L_1 and L_2 are $\vec{u_1} = (2,3,-1)$ and $\vec{u_2} = (-1,2,4)$, respectively. Since these vectors are not scalar multiples of each other, L_1 and L_2 are not parallel lines.

We must now show that L_1 and L_2 do not intersect. At any point of intersection, the system

$$\left.\begin{aligned} 3(x+2) &= 2(y-1) \\ 2(x-1) &= -(y+1) \\ -(x+2) &= 2(z+1) \\ 4(x-1) &= -(z-2) \end{aligned}\right\}$$

would have a solution in x, y, z. The first two equations give $x = \frac{6}{7}$, $y = \frac{19}{7}$. But then the third equation would give $z = -\frac{11}{7}$, while the fourth equation would give $z = \frac{66}{7}$. Since the system is inconsistent, L_1 and L_2 can have no point of intersection.

b) Write down the equation of the plane $\mathcal{P}$ which contains the line L_2 such that the line L_1 is parallel to the plane $\mathcal{P}$.

Since the plane P contains L_2 and L_1 is parallel to it, the normal vector for the plane can be taken to be $\vec{u_1} \times \vec{u_2}$. Calculating, we find $\vec{u_1} \times \vec{u_2} = 14i - 7j + 7k$. Simplifying, we take the normal vector to be $2i - j + k$. Since P contains L_2, it contains the point $(1,-1,2)$ in particular. Therefore, the equation of the plane P is

$$2(x-1) - (y+1) + (z-2) = 0,$$

or

$$2x - y + z - 5 = 0$$

c) Find the distance between the line L_1 and the plane $\mathcal{P}$ found in part (b).

The distance between the line L_1 and the plane P is equal to the distance from any point of L_1 to P (because L_1 is parallel to P). Taking the point on L_1 to be $(-2,1,-1)$, the distance from $(-2,1,-1)$ to P is

$$\frac{|2.(-2) - 1 + (-1) - 5|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{11}{\sqrt{6}}$$

2. (13+7 pts.) Let $L_1 : (x, y, z) = (-2 + 2t, 3 + 3t, 6t)$, $t \in \mathbb{R}$ and $L_2 : (x, y, z) = (-1 + 2t, 2 + 3t, -2 + 6t)$, $t \in \mathbb{R}$ be two lines in $\mathbb{R}^3$.

a) Calculate the distance between L_1 and L_2.

Direction vectors for the lines L_1 and L_2 are the same, namely $\vec{v} = (2,3,6)$. $P_1(-2,3,0)$ is a point on L_1 but $P_1 \notin L_2$. Hence $L_1 \parallel L_2$. Moreover $P_2(-1,2,-2)$ is a point on L_2.

Set $\vec{u} = \vec{P_1} - \vec{P_2}$. Then $\vec{u} = (-1,1,2)$ and

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 3 & 6 \end{vmatrix} = (0, 10, -5).$$

Therefore

$$|L_1 L_2| = |P_1 L_2| = \frac{|\vec{u} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{100+25}}{\sqrt{4+9+36}} = \sqrt{\frac{125}{49}} = \frac{5\sqrt{5}}{7}.$$

b) Find an equation of the plane $\mathcal{P}$ which contains both of the lines L_1 and L_2.

A normal vector for $\mathcal{P}$ is $\vec{N} = \vec{u} \times \vec{v} = (0, 10, -5)$ and a point on $\mathcal{P}$ is $P_1(-2,3,0)$ since $P_1 \in L_1 \subseteq \mathcal{P}$. An equation of $\mathcal{P}$ is given by

$$10(y-3) - 5z = 0$$

or equivalently

$$2y - z = 6.$$

3. (10+10 pts.) a) Show that the five points $A(1, 1, 2)$, $B(3, 5, 4)$, $C(0, -3, 5)$, $D(3, 7, 0)$, and $E(4, 9, 1)$ are coplanar (that is, they all lie on a plane).

$\vec{AB} = (2,4,2)$, $\vec{AC} = (-1,-4,3)$, $\vec{N} = \vec{AB} \times \vec{AC} = (20,-8,-4) \neq 0$

Let $\mathcal{P}$ be the (unique) plane such that $A, B, C \in \mathcal{P}$.

$\mathcal{P}$: $\vec{N} \cdot \vec{P} = \vec{N} \cdot \vec{A}$: $20x - 8y - 4z = 4$

$E = (4,9,1)$ $\quad 20.4 - 8.9 - 4 = 4$ $\quad E \in \mathcal{P}$

$D = (3,7,0)$ $\quad 20.3 - 8.7 - 4.0 = 4$ $\quad D \in \mathcal{P}$

A, B, C, D, E are coplanar.

3.b) Write down the parametric equations of the line L which is the intersection of the two planes $2x - 3y + 5z = 15$ and $x + 4y + 3z = 2$.

$$\left.\begin{array}{l} P_1:\ x + 4y + 3z = 2 \\ P_2:\ 2x - 3y + 5z = 15 \end{array}\right\} \Leftrightarrow \begin{array}{l} x + 4y + 3z = 2 \\ -11y - z = 11 \end{array}$$

Take $z = 0$

$P_0(6, -1, 0) \in L$

Direction vector of the line can be taken $\vec{u} = \vec{N_1} \times \vec{N_2} = (-29, -1, 11)$

$L:\ \vec{P} = \vec{P_0} + t.\vec{u}$

$(x, y, z) = (6, -1, 0) + t(-29, -1, 11)$

$$L: \begin{cases} x = 6 - 29t \\ y = -1 - t \\ z = 11t \end{cases},\ t \in \mathbb{R}$$

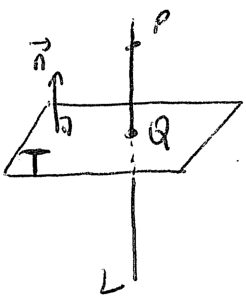
or find P_1 for setting z any value other than zero. Then

$L:\ \vec{P} = \vec{P_0} + t.\overrightarrow{P_0P_1}$

4. (10+10 pts.) Let $P = (1, 3, 4)$ and $\mathcal{T}$ be the plane given by $x - 2y + 5z = 5$.

a) Find the point $Q \in \mathcal{T}$ which is closest to P among all points of $\mathcal{T}$.

Let L be the line through P such that $L \perp T$.



$\vec{n} = (1, -2, 5)$ is a normal vector of T. We can take $\vec{v} = \vec{n}$ as a direction vector of L since $L \perp T$.

Q: closest point to P among all points of $T \Rightarrow Q = L \cap T$.

$L:\ (x, y, z) = \vec{P} + t\vec{v}$, $t \in \mathbb{R}$

$(x, y, z) = (1, 3, 4) + t(1, -2, 5)$, $t \in \mathbb{R}$

Then $Q = (x, y, z) \in L \Rightarrow x = 1 + t,\ y = 3 - 2t,\ z = 4 + 5t$

$Q \in T \Rightarrow x - 2y + 5z = 5 \Rightarrow (1+t) - 2(3 - 2t) + 5(4 + 5t) = 5 \Rightarrow t = -\frac{1}{3}$

$t = -\frac{1}{3} \Rightarrow Q = (x, y, z) = (1+t, 3-2t, 4+5t) = \left(\frac{2}{3}, \frac{11}{3}, \frac{7}{3}\right)$ $\qquad Q = \left(\frac{2}{3}, \frac{11}{3}, \frac{7}{3}\right)$

b) Write down a vector equation of the line L_1 which passes through P such that L_1 is parallel to the plane $\mathcal{T}$ and L_1 intersects the line L_2 at a point, where L_2 is the line given by $L_2: \dfrac{x-1}{2} = \dfrac{y-6}{5} = \dfrac{z-1}{3} = t$

Let L_1 and L_2 intersect at $P_2 = (x, y, z)$. $P_2 = L_1 \cap L_2 \Rightarrow P_2 \in L_2$, so $P_2 = (x, y, z) = (1+2t, 6+5t, 1+3t)$ for some $t \in \mathbb{R}$

Then $\vec{v} = \overrightarrow{PP_2} = \vec{P_2} - \vec{P}$ is a direction vector of L_1,

$\vec{v} = \vec{P_2} - \vec{P} = (1+2t, 6+5t, 1+3t) - (1, 3, 4) = (2t, 3+5t, -3+3t)$

$L_1 \parallel T \Rightarrow \vec{v} \perp \vec{n} = (1, -2, 5)$ normal vector of T.

Thus $\vec{v} \cdot \vec{n} = 0 \Rightarrow (2t, 3+5t, -3+3t) \cdot (1, -2, 5) = 0 \Rightarrow 2t - 2(3+5t) + 5(-3+3t) = 0$

$\Rightarrow \boxed{t = 3}$

$t = 3 \Rightarrow P_2 = (1 + 2.3, 6 + 5.3, 1 + 3.3) \Rightarrow \vec{P_2} = (7, 21, 10)$

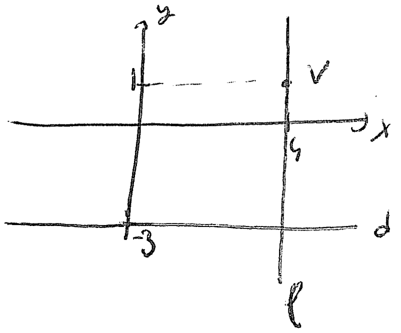
$\vec{v} = \overrightarrow{PP_2} = \vec{P_2} - \vec{P} = (7, 21, 10) - (1, 3, 4) = (6, 18, 6)$

vector eq. of L_1: $(x, y, z) = (1, 3, 4) + t(6, 18, 6)$, $t \in \mathbb{R}$

$(x, y, z) = (1 + 6t, 3 + 18t, 4 + 6t)$, $t \in \mathbb{R}$

5. (4×5 **pts.**) Let S be the parabola with vertex at $V(4,1)$ which has the line $d = \{(x,y) \in \mathbb{R}^2 \mid y = -3\}$ as its directrix.

a) Find the equation of the axis ℓ of S.



$\ell \perp d$, $V \in \ell$

d: $y=-3$ (horizontal line)

$d \perp \ell \Rightarrow \ell$ is a vertical line

Thus ℓ: $x=4$

b) Find the point of intersection G of d and ℓ.

$\left.\begin{array}{l} d: y=-3 \\ \ell: x=4 \end{array}\right\}$ $\ell \cap d = G = (4,-3)$

c) Find the focus F of S.

$e=1$ for a parabola , $\vec{V} = \dfrac{\vec{F} + e.\vec{G}}{1+e} = \dfrac{\vec{F}+\vec{G}}{2} \Rightarrow 2.\vec{V} = \vec{F} + \vec{G}$

$\vec{F} = 2\vec{V} - \vec{G}$

$\vec{F} = 2.(4,1) - (4,-3) = (4,5)$

$\vec{F} = (4,5)$

d) Find an equation, in coordinate form, of S.

$P=(x,y)$, $P \in S \Leftrightarrow \dfrac{|PF|}{|Pd|} = e = 1 \Leftrightarrow |PF| = |Pd|$

d: $y+3=0$

$|Pd| = \dfrac{|y+3|}{\sqrt{0^2+1^2}} = |y+3|$

$|PF| = \sqrt{(x-4)^2 + (y-5)^2}$

$\Leftrightarrow \sqrt{(x-4)^2+(y-5)^2} = |y+3|$

$(x-4)^2 + (y-5)^2 = (|y+3|)^2 = (y+3)^2$

$x^2 - 8x + 16 + y^2 - 10y + 25 = y^2 + 6y + 9$

$\boxed{x^2 - 8x - 16y + 32 = 0}$

Equation of S.