M E T U
Department of Mathematics

	Analytic Geometry Midterm I	
Code : *Math 115* Acad. Year : *2017-2018* Semester : *Fall* Coordinator: *E. Coskun* Date : *09.11.2017* Time : *17.40* Duration : *110 minutes*	Last Name : Name : Student No : Department : Signature :	5 Questions on 4 Pages Total 100 Points

1	2	3	4	5		

1. (20 pts) Consider the points $A(3,1)$ and $B(7,-5)$ in the Cartesian plane.

a. Find the equation of the line ℓ_1 which passes through A and B.

$$m = \frac{-5-1}{7-3} = -\frac{3}{2}$$

$y - y_0 = m(x - x_0)$ $y - 1 = -\frac{3}{2}(x-3)$ or $y+5 = -\frac{3}{2}(x-7)$ or $3x + 2y - 11 = 0$

b. Find the midpoint M of the line segment AB.

$$\left(\frac{3+7}{2}, \frac{1-5}{2}\right) = (5, -2)$$

c. Find the equation of the perpendicular bisector ℓ_2 of the line segment AB, i.e. the line that intersects the line segment AB at its midpoint with a right angle.

$$m_2 = -\frac{1}{m} = \frac{2}{3}$$

$y + 2 = \frac{2}{3}(x-5)$ or $-2x + 3y + 16 = 0$

2. (20 pts) Consider the polar equation $r = 2sin(2\theta)$ for $0 \leq \theta \leq \pi$.

a. Write the values of r for the given values of θ in the table.

θ	0	$\pi/12$	$\pi/8$	$\pi/6$	$\pi/4$	$\pi/3$	$3\pi/8$	$5\pi/12$	$\pi/2$	$7\pi/12$	$5\pi/8$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$7\pi/8$	$11\pi/12$	π
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2	$-\sqrt{3}$	$-\sqrt{2}$	-1	0

$2\sin 0 = 0$

$2 \cdot \sin(2 \cdot \frac{\pi}{12}) = 2 \cdot \sin(\frac{\pi}{6}) = 1$

$2 \cdot \sin(2 \cdot \frac{\pi}{8}) = 2 \cdot \sin\frac{\pi}{4} = \sqrt{2}$

$2\sin(2 \cdot \frac{\pi}{6}) = 2\sin\frac{\pi}{3} = \sqrt{3}$

$2 \cdot \sin(2 \cdot \frac{\pi}{4}) = 2\sin(\frac{\pi}{2}) = 2$

$2\sin\frac{2\pi}{3} = \sqrt{3}$

$2\sin\frac{3\pi}{4} = \sqrt{2}$

$2\sin\frac{5\pi}{6} = 1$

$2\sin\pi = 0$

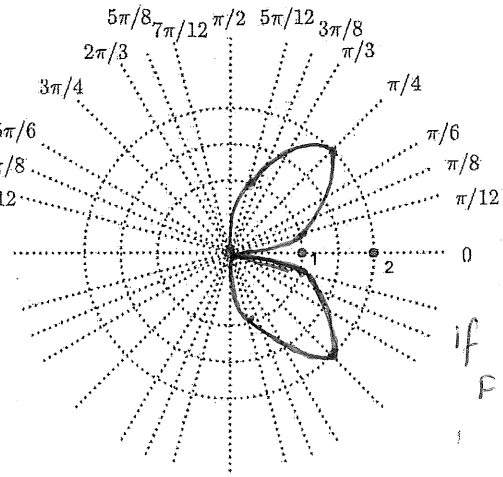
$2\sin\frac{7\pi}{6} = 2\sin\frac{11\pi}{6} = -1$

$2\sin\frac{5\pi}{4} = 2\sin\frac{7\pi}{4} = -\sqrt{2}$

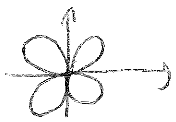
$2\sin\frac{4\pi}{3} = 2\sin\frac{5\pi}{3} = -\sqrt{3}$

$2\sin\frac{3\pi}{2} = -2$

b. Sketch the graph of the equation $r = 2\sin(2\theta)$ for $0 \leq \theta \leq \pi$.



Note that if we consider all θ values, not just $\theta \in [0, \pi]$, then the full graph is



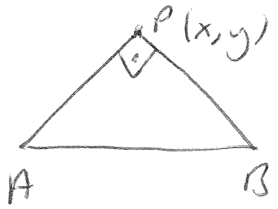
Note that $\theta \in [\frac{\pi}{2}, \pi] \Rightarrow r = 2\sin 2\theta \leq 0$

Therefore, points P on the graph of $r = 2\sin 2\theta$ for which $\theta \in [\frac{\pi}{2}, \pi]$ are in the 4th quadrant $(x \geq 0,\ y \leq 0)$.

if $F(r, \theta) = r - 2\sin 2\theta = 0$, then

$F(-r, \pi - \theta) = -r - 2\sin(2(\pi - \theta))$
$= -r - 2\sin(2\pi - 2\theta)$
$= -r - 2\sin(-2\theta) = -(r - 2\sin 2\theta)$
$= 0$. Thus, graph is symmetric with respect to x-axis.

3. (20 pts) Consider the points $A(-2, 5)$, $B(2, 1)$ and the line L with equation $y = 2x + 1$ in the Cartesian plane. Find all points P on L such that the triangle APB is a right triangle (with right angle being at vertex P).



$P(x, y)$, $y = 2x + 1$

$\overrightarrow{AP} = (x - (-2), y - 5) = (x + 2, y - 5)$

$\overrightarrow{BP} = (x - 2, y - 1) = (x - 2, y - 1)$

$\Rightarrow \overrightarrow{AP} \cdot \overrightarrow{BP} = (x+2)(x-2) + (y-1)(y-5) = 0$

$= (x+2)(x-2) + (2x+1-1)(2x+1-5) = 0$

$= (x+2)(x-2) + 2x \cdot 2(x-2) = 0$

$= (x-2)(x+2+4x) = 0$

$= (x-2)(5x+2) = 0$

$\Rightarrow x = 2 \Rightarrow y = 5$

$x = -\frac{2}{5} \Rightarrow y = \frac{1}{5}$

So $P(2, 5)$ and $P(-\frac{2}{5}, \frac{1}{5})$

4. (20 pts)

a. Show that the points $A(-1,3)$, $B(3,11)$ and $C(5,15)$ are collinear (i.e. they lie on a line in the Cartesian plane).

l_{AB} : $\frac{y-3}{x+1} = \frac{11-3}{3-(-1)} = 2$; $-2x+y-5=0$

$C \in l_{AB}$

$-2.5+15-5=0$

b. Find a unit vector $\vec{u}$ that has the same direction as $8\vec{i} - \vec{j} + 4\vec{k}$.

$|8i - j + 4k| = \sqrt{64+1+16} = 9$

$\vec{u} = \frac{1}{9}(8,-1,4)$

c. Find the angle between the vectors $\vec{v} = \vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{w} = \vec{i} - \vec{k}$.

$v \cdot w = 1+2=3$

$|v|=3$ $|w|=\sqrt{2}$

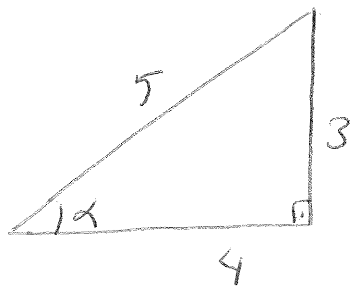
$\cos\theta = \frac{v \cdot w}{|v||w|} = \frac{1}{\sqrt{2}}$ $0 \leq \theta \leq \pi$

$\theta = \frac{\pi}{4}$

5. (20 pts) Assume that the $\bar{x}\bar{y}$ coordinate system is obtained from the xy-coordinate system by a rotation through an angle $\alpha = \tan^{-1}(3/4)$ in counter-clockwise direction.

a. Find $\cos(\alpha)$ and $\sin(\alpha)$.

$$\cos(\alpha) = \frac{4}{5} \quad \text{and} \quad \sin(\alpha) = \frac{3}{5}$$



b. Write x and y in terms of $\bar{x}$ and $\bar{y}$.

$$x = \bar{x}\cos(\alpha) - \bar{y}\sin(\alpha) = \frac{4\bar{x} - 3\bar{y}}{5}$$

$$y = \bar{x}\sin(\alpha) + \bar{y}\cos(\alpha) = \frac{3\bar{x} + 4\bar{y}}{5}$$

c. Let L be the line with xy-equation $3x - 4y = 0$. Find the equation of L in the $\bar{x}\bar{y}$ coordinate system.

Since L is the line with xy-equation $3x-4y=0$, L has slope $3/4$ which is equal to $\tan(\alpha)$! Therefore, when xy-coordinate system is rotated through the angle $\alpha = \tan^{-1}(3/4)$ to obtain the $\bar{x}\bar{y}$-coordinate system the $\bar{x}$-axis and the line L coincide. Hence the $\bar{x}\bar{y}$-equation of L is $\bar{y}=0$.

OR

Substitute x and y found in (b) into $3x-4y=0$ to obtain:

$$3\left(\frac{4\bar{x}-3\bar{y}}{5}\right) - 4\left(\frac{3\bar{x}+4\bar{y}}{5}\right) = \frac{12\bar{x}-9\bar{y}-12\bar{x}-16\bar{y}}{5} = 0.$$

Thus an equation of L in the $\bar{x}\bar{y}$-coordinates is given by $\bar{y}=0$.

d. Let P be the point whose xy-coordinates are $(5, 10)$. Find the $\bar{x}\bar{y}$ coordinates of P.

$$\left.\begin{aligned} 5 &= \frac{4\bar{x}-3\bar{y}}{5} \\ 10 &= \frac{3\bar{x}+4\bar{y}}{5} \end{aligned}\right\} \text{ imply } \quad \begin{aligned} 4\bar{x}-3\bar{y} &= 25 \\ 3\bar{x}+4\bar{y} &= 50 \end{aligned}$$

Then we have $25\bar{x} = 250$, i.e. $\bar{x} = 10$ and $\bar{y} = 5$.

The $\bar{x}\bar{y}$-coordinates of P are $(10, 5)$.

e. Use part (d) to find the distance from the point P to the $\bar{y}$-axis.

As we can see from the figure the distance from P to the $\bar{y}$-axis is the $\bar{x}$-coordinate of P, i.e. it is 10.