M E T U
Department of Mathematics

	Analytic Geometry Final Exam	
Code : *Math 115* Acad. Year : *2017-2018* Semester : *Fall* Coordinator: *E. Coskun* Date : *13.1.2018* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Student No : Department : Signature : 5 Questions on **4** Pages Total **100** Points	

1	2	3	4	5		

1. **(5+5+5+5 pts.)** Consider the hyperbola whose foci are $F(4,4)$, $F'(-4,-4)$ and eccentricity is $e=2$.

(a) Find the center C.

$C(0,0)$.

(b) Write an equation for the axis ℓ of the hyperbola.

ℓ; $y=x$

(c) Find the intersections G and G' of ℓ with the directrices d and d', respectively.

$G = \frac{1}{e^2} F = \frac{1}{4}(4,4) = (1,1)$

G' and G are symmetric with respect to the origin, so $G' = (-1,-1)$.

(d) Find the xy-equation of the hyperbola.

$d: \ \frac{y-1}{x-1} = -1 \Rightarrow y+x-2=0$

$\frac{|PF|^2}{|Pd|^2} = e^2 \Rightarrow \frac{(x-4)^2+(y-4)^2}{\left(\frac{y+x-2}{\sqrt{1^2+1^2}}\right)^2} = 2^2$

$\Rightarrow y^2+x^2+4xy-24=0$.

2. **(3+2+3+6+6 pts.)** Let S be a surface with the equation $4(x-1)^2+9(z-3)^2=(y-4)^2$.

(a) Identify the surface S. What type of a quadratic surface is S? (Explain).

Solution : $4(x-1)^2+9(z-3)^2=(y-4)^2$ can be rewritten as $\dfrac{(x-1)^2}{9}+\dfrac{(z-3)^2}{4}=\dfrac{(y-4)^2}{36}$ which is an equation of an elliptic cone.

(b) Describe the plane section of S by the plane $\mathcal{P}_1 : y=4$, i.e. find $S\cap\mathcal{P}_1$.

Solution : Insert $y=4$ into $\dfrac{(x-1)^2}{9}+\dfrac{(z-3)^2}{4}=\dfrac{(y-4)^2}{36}$ to get $\dfrac{(x-1)^2}{9}+\dfrac{(z-3)^2}{4}=0$ which implies

$x=1,\ z=3$. Thus $S\cap\mathcal{P}_1=\{(1,4,3)\}$.

(c) Describe the plane section of S by the plane $\mathcal{P}_2 : x=1$.

Solution : Insert $x=1$ into $4(x-1)^2+9(z-3)^2=(y-4)^2$ to get $z=3\pm\frac{1}{3}(y-4)$. In other words $S\cap\mathcal{P}_2$ consists of two intersecting lines $z=\dfrac{5+y}{3}$ and $z=\dfrac{13-y}{3}$.

Write the type and find the eccentricity e of the conic C which is the intersection of the given surface with each of the following planes (In each case write the equation of the conic in its simplest form):

(d) $y=6$

Solution : Insert $y=6$ into $4(x-1)^2+9(z-3)^2=(y-4)^2$ which implies $(x-1)^2+\dfrac{(z-3)^2}{4/9}=1$. This is an ellipse equation with $a=1$ and $b=2/3$. Since $b=a\sqrt{1-e^2}$ we have $e=\sqrt{5}/3$.

(e) $z=4$

Solution : Insert $z=4$ into $4(x-1)^2+9(z-3)^2=(y-4)^2$ which implies $\dfrac{(y-4)^2}{9}-\dfrac{(x-1)^2}{9/4}=1$. This is a hyperbola equation with $a=3$ and $b=3/2$. Since $b=a\sqrt{e^2-1}$ we have $e=\sqrt{5}/2$.

3. **(14+6 pts.)** Let S be a conic with the equation $2x^2+2\sqrt{2}xy+3y^2=1$.

(a) Find $\cos\alpha$ and $\sin\alpha$ such that $0<\alpha<\pi/2$ and when xy-coordinate system is rotated by α radians to obtain $\bar{x}\bar{y}$-coordinate system, the equation of S has no $\bar{x}\bar{y}$-term. Moreover write down x and y in terms of $\bar{x}$ and $\bar{y}$.

$A=2,\quad 2B=2\sqrt{2},\quad C=3.\qquad \cot(2\alpha)=\dfrac{A-C}{2B}=\dfrac{2-3}{2\sqrt{2}}=\dfrac{-1}{2\sqrt{2}},\quad 0<\alpha<\dfrac{\pi}{2}$

$\sin^2(2\alpha)=\dfrac{1}{\csc^2(2\alpha)}=\dfrac{1}{1+\cot^2(2\alpha)}\Rightarrow \sin(2\alpha)=\dfrac{2\sqrt{2}}{3}$

(since $0<\alpha<\frac{\pi}{2}$, $\sin(2\alpha)\geq 0$)

$\cos(2\alpha)=\cot(2\alpha).\sin(2\alpha)=\dfrac{-1}{3}$

$0<\alpha<\frac{\pi}{2}\Rightarrow \sin\alpha>0,\ \cos\alpha>0 \Rightarrow \sin\alpha=\sqrt{\dfrac{1-\cos 2\alpha}{2}}=\dfrac{\sqrt{2}}{\sqrt{3}}$

$\cos\alpha=\sqrt{\dfrac{1+\cos 2\alpha}{2}}=\dfrac{1}{\sqrt{3}}$

Rotation formulas :

$x=\cos\alpha.\bar{x}-\sin\alpha.\bar{y}\Rightarrow x=\dfrac{\bar{x}-\sqrt{2}\,\bar{y}}{\sqrt{3}}$

$y=\sin\alpha.\bar{x}+\cos\alpha.\bar{y}\Rightarrow y=\dfrac{\sqrt{2}\,\bar{x}+\bar{y}}{\sqrt{3}}$

(b) Write down the equation of S in terms of the $(\bar{x}, \bar{y})$ coordinates.

$$2x^2 + 2\sqrt{2}\,xy + 3y^2 = 1 \iff 2\cdot\left(\frac{\bar{x} - \sqrt{2}\,\bar{y}}{\sqrt{3}}\right)^2 + 2\sqrt{2}\left(\frac{\bar{x} - \sqrt{2}\bar{y}}{\sqrt{3}}\right)\left(\frac{\sqrt{2}\,\bar{x} + \bar{y}}{\sqrt{3}}\right)$$

$$+ 3\left(\frac{\sqrt{2}\,\bar{x} + \bar{y}}{\sqrt{3}}\right)^2 = 1$$

$$\iff \frac{1}{3}\left(12\,\bar{x}^2 + 0.\bar{x}\bar{y} + 3\bar{y}^2\right) = 1$$

$$\iff 4\bar{x}^2 + \bar{y}^2 = 1$$

4. **(6+7+7 pts.)** Let ℓ be a line and $\mathcal{P}$ be a plane in 3-space. P and Q are said to be symmetric (partners of each other) about the line ℓ if ℓ is a perpendicular bisector of the segment $[PQ]$. P and Q are said to be symmetric (partners of each other) about the plane $\mathcal{P}$ if $\mathcal{P}$ is perpendicular to the segment $[PQ]$ and bisects it. Let $P(4, 2, 8)$ be a point. Find the symmetric partner Q of P with respect to

(a) the point $M(2, 4, 6)$ Let $Q(a,b,c)$.

M is the midpoint of PQ : $\frac{a+4}{2} = 2$, $\frac{b+2}{2} = 4$, $\frac{c+8}{2} = 6$

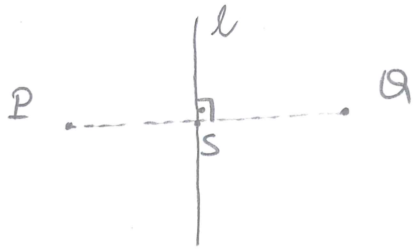
$\Rightarrow Q(0,6,4)$

(b) the plane $\mathcal{P} : x = 10$

$Q(x_0, 2, 8)$ and $\frac{x_0 + 4}{2} = 10 \Rightarrow x_0 = 16$

So, $Q(16, 2, 8)$.

(c) the line $\ell : (x, y, z) = (4, 2, 4) + t\underbrace{(1, 1, 0)}_{v}$ for $t \in \mathbb{R}$.



Find $S \in \ell$: $\overrightarrow{PS} \perp \vec{v}$

$S(4+t, 2+t, 4)$, $\overrightarrow{PS} = (t, t, -4)$

$\overrightarrow{PS} \perp \vec{v} \iff \overrightarrow{PS} \cdot \vec{v} = 0$

$\iff t + t = 0$

$\iff t = 0$.

So, $S(4, 2, 4)$

$Q(a,b,c)$, now S is the midpoint of PQ.

So, $Q(4, 2, 0)$.

5. (10+10 pts.)
(a) Write the equation of the ellipse with foci $F_1(3,0)$, $F_2(-3,0)$ and minor axis (diameter) 8 units long.

Solution: Note that the foci lie on the x-axis and they are symmetric with respect to the origin. Hence, the equation of the ellipse should be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with $b=a\sqrt{1-e^2}$ and the foci at the points $(ae,0)$ and $(-ae,0)$.

Since the minor axis is 8 units long, we have $b=4$. Then, $ae=3$ and $a\sqrt{1-e^2}=4$. This gives

$$\frac{e}{\sqrt{1-e^2}}=\frac{3}{4}.$$

Solving, we get $e=3/5$ and hence $a=5$. Therefore, the equation of the ellipse is

$$\boxed{\frac{x^2}{25}+\frac{y^2}{16}=1}$$

(b) Find the value(s) of m so that the plane $\mathcal{P}: x+z=m$ touches (i.e. intersects at one point) the sphere $x^2+y^2+z^2=4$.

Solution: The equation of the plane $\mathcal{P}$ gives $z=m-x$. In order for $\mathcal{P}$ to intersect the sphere $x^2+y^2+z^2=4$ at one point, the equation

$$x^2+y^2+(m-x)^2=4$$

must have *exactly one* solution. Expanding, we get

$$x^2+y^2+m^2-2mx+x^2=4,$$

and simplifying, we get

$$2x^2-2mx+y^2=4-m^2.$$

After completing the square, we get

$$2(x-m/2)^2+y^2=4-m^2/2.$$

In order for this equation to have *exactly one* solution, the number on the right must be zero. Hence,

$$m^2=8,$$

and this gives

$$\boxed{m=\pm2\sqrt{2}}$$