Math 119 2020-1 Recitation Week 9

① Find all points on the curve $y = 1 + x^{3/2}$ farthest from or closest to the point $(8,1)$.

Solution :

Let (a,b) be a point on $y = 1 + x^{3/2}$. So, $b = 1 + a^{3/2}$. Then the distance between (a,b) and $(8,1)$ is

$$d = \sqrt{(a-8)^2 + (b-1)^2} \underset{b = 1 + a^{3/2}}{=} \sqrt{(a-8)^2 + a^3}$$

$$\Rightarrow d^2 = (a-8)^2 + a^3 \underset{\text{call it}}{=} f(a)$$

Minimizing (Maximizing) d is equivalent to minimizing (maximizing) $f(a)$

$$f'(a) = 3a^2 + 2(a-8) = 3a^2 + 2a - 16$$
$$= (3a+8)(a-2) = 0$$

$$\Rightarrow a = 2, \quad a = -\frac{8}{3}$$

a		$-\frac{8}{3}$		2	
$f'(a)$	$+$		$-$		$+$
$f(a)$	$\nearrow$	local max.	$\searrow$	local min.	$\nearrow$

Since the domain of $y = 1 + x^{3/2}$ is $[0, \infty)$, we do <u>not</u> need to consider the region before 0.

So, $(2, 1 + 2^{3/2})$ is on the curve and is closest to $(8,1)$.

$$\lim_{a \to \infty} f(a) = \lim_{a \to \infty} \left((a-8)^2 + a^3\right) = \infty$$

So, there is no point on $y = 1 + x^{3/2}$ which is farthest from $(8,1)$.

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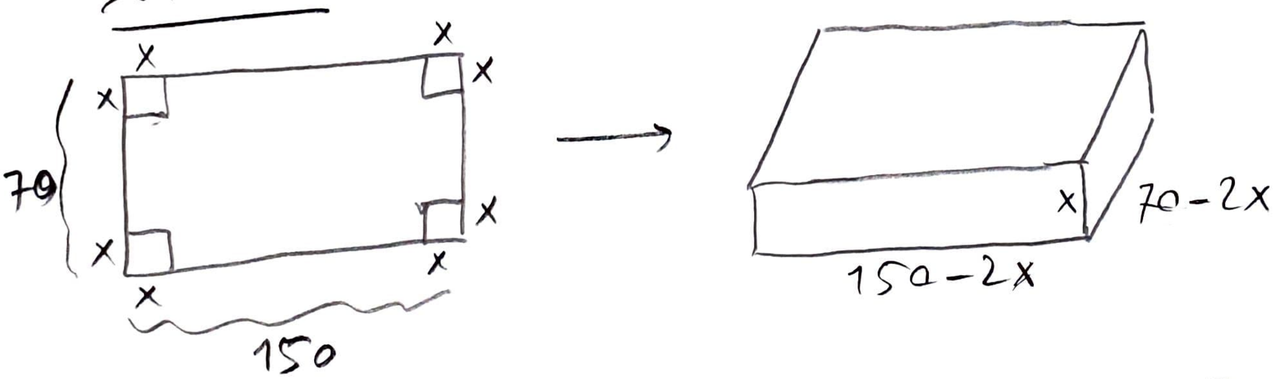
② A box is to be made from a rectangular sheet of cardboard 70 cm x 150 cm by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. (The box has no top.) What is the largest volume of the box?

Solution



$V(x) = x(70-2x)(150-2x)$ on $[0, 35]$

$V(x) = 0 \Rightarrow x = 0,\ x = 35,\ x = 75$ not possible since $0 \le x \le 70$

x	0		35		70
$V(x)$		$+$		$-$	

since $V(x) \geq 0$

V is a polynomial $\Longrightarrow$ V is continuous on $[0,35]$ (closed, bounded)

By Extreme Value Theorem, V has an absolute maximum on $[0,35]$.

$$V'(x) = (70-2x)(150-2x) + x(-2)(150-2x) + x(70-2x)(-2)$$

$$\Rightarrow V'(x) = 4(3x-175)(x-15) = 0$$

$$\Rightarrow \begin{array}{l} x = 15 \in [0,35] \\ x = \frac{175}{3} \notin [0,35] \end{array}$$

So, at $x = 15$, V has a critical point.

$V(15) = 15(70-30)(150-30) = 72000$

$V(0) = 0$

$V(35) = 0$

So, the largest volume of box is 72000.

(3) Let $f(x)$ be a differentiable function where $f(-8) = -2$ and $f'(-8) = \frac{1}{12}$.
Approximate the value of $f(-7{,}98)$.

Solution

The linearization of f about $x = a$ is the function

$$L(x) = f(a) + f'(a)(x-a).$$

$$a = -8 \Rightarrow f(x) \approx L(x) = f(-8) + f'(-8)(x+8)$$
$$= -2 + \frac{1}{12}(x+8)$$

$$\Rightarrow f(-7{,}98) \approx -2 + \frac{0{,}02}{12} = -2 + \frac{1}{600} = -\frac{1199}{600}$$

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(4) Use the linearization of a suitable function about a suitable value, approximate

$$\frac{1}{\sqrt[4]{0{,}0083}}.$$

Solution : let $f(x) = \frac{1}{\sqrt[4]{x}} = x^{-1/4}$

The linearization of f about $x = a$ is

$$L(x) = f(a) + f'(a)(x-a).$$

let $a = 0{,}0081$. Then

$$L(x) = \underbrace{f(0,0081)}_{\substack{\| \\ (0,3)^{-1} = \frac{10}{3}}} + f'(0,0081)\,(x - 0,0081)$$

$$f'(x) = -\frac{1}{4}\,x^{-5/4} \Rightarrow f'(0,0081) = -\frac{1}{4}\,\frac{1}{(0,3)^5}$$

$$\Rightarrow L(x) = \frac{10}{3} - \frac{1}{4}\cdot\frac{1}{(0,3)^5}\,(x - 0,0081)$$

$$\Rightarrow f(0,0083) \approx L(0,0083) = \frac{10}{3} - \frac{1}{4}\,\frac{1}{(0,3)^5}\,(0,0002)$$

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(5) Calculate the lower and upper Riemann sums for $f(x) = 4 - |x-3|$ corresponding to partition P of the given interval

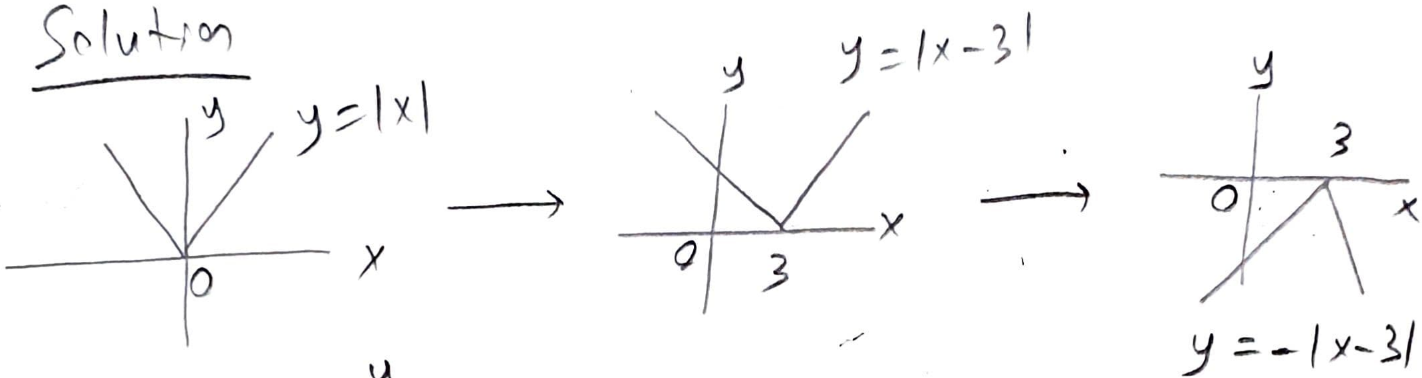
(a) into four subintervals of equal length on $[2,5]$

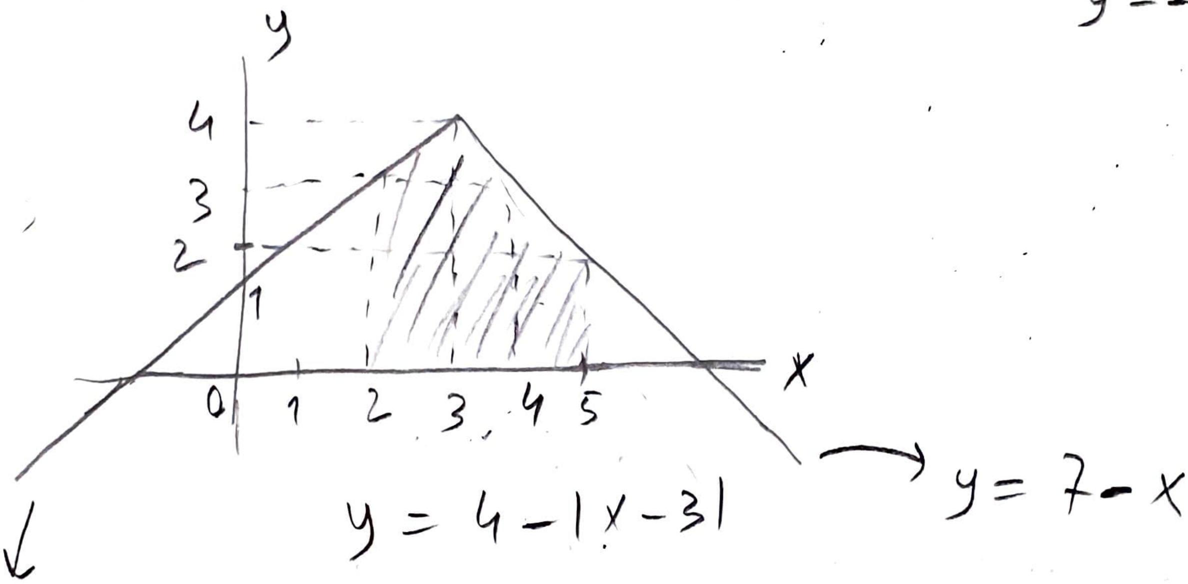
(b) into six subintervals of equal length on $[2,5]$

(c) into n subintervals on $[2,3]$ and $[3,5]$

(d) Evaluate the limits you obtained in part (c) as $n \to \infty$ and compare the obtained results in (a) and (b).

Solution





f is increasing on $[2,3]$

f is decreasing on $[3,5]$

(a) $f(x) = 4 - |x-3|$ on $[2,5]$

# of intervals is $n = 4$

length of each subinterval $\Delta x_i = \dfrac{x_4 - x_0}{4} = \dfrac{5-2}{4} = \dfrac{3}{4}$

$$\Rightarrow P_4 = \left\{ 2,\ 2+\frac{3}{4},\ 2+2\cdot\frac{3}{4},\ 2+3\cdot\frac{3}{4},\ 2+4\cdot\frac{3}{4} \right\}$$

$$= \left\{ \underset{x_0}{2},\ \underset{x_1}{\frac{11}{4}},\ \underset{x_2}{\frac{14}{4}},\ \underset{x_3}{\frac{17}{4}},\ \underset{x_4}{5} \right\}$$

$f(x_0) = 3,\ f(x_1) = \frac{15}{4},\ f(x_2) = \frac{14}{4},\ f(x_3) = \frac{11}{4},\ f(x_4) = 2$

$$L(f,P_4)=\sum_{i=1}^{4} f(x_i^*)\,\Delta x_i = \frac{3}{4}\sum_{i=1}^{4} f(x_i^*)$$

$$=\frac{3}{4}\Big(f(\underset{x_0}{\underset{\parallel}{x_1^*}})+f(\underset{x_2}{\underset{\parallel}{x_2^*}})+f(\underset{x_3}{\underset{\parallel}{x_3^*}})+f(\underset{x_4}{\underset{\parallel}{x_4^*}})\Big)$$

$$=\frac{3}{4}\Big(f(2)+f\Big(\frac{14}{4}\Big)+f\Big(\frac{17}{4}\Big)+f(5)\Big)$$

$$=\frac{3}{4}\Big(3+\frac{14}{4}+\frac{11}{4}+2\Big)=\frac{135}{16}$$

$$U(f,P_4)=\sum_{i=1}^{4} f(x_i^*)\,\Delta x_i = \frac{3}{4}\sum_{i=1}^{4} f(x_i^*)$$

$$=\frac{3}{4}\Big(f(\overset{x_1}{\overset{\parallel}{x_1^*}})+f(\overset{3}{\overset{\parallel}{x_2^*}})+f(\overset{x_2}{\overset{\parallel}{x_3^*}})+f(\overset{x_3}{\overset{\parallel}{x_4^*}})\Big)$$

$$=\frac{3}{4}\Big(f\Big(\frac{11}{4}\Big)+f(3)+f\Big(\frac{14}{4}\Big)+f\Big(\frac{17}{4}\Big)\Big)$$

$$=\frac{3}{4}\Big(\frac{15}{4}+4+\frac{14}{4}+\frac{11}{4}\Big)=\frac{168}{16}$$

(b) $f(x)=4-|x-3|$ on $[2,5]$

\# of subintervals is $n=6$

Length of each subinterval $\Delta x_i=\dfrac{x_6-x_0}{6}=\dfrac{5-2}{6}=\dfrac{1}{2}$

$$\Rightarrow P_6=\Big\{\underset{x_0}{\underset{\parallel}{2}},\underset{x_1}{\underset{\parallel}{\tfrac{5}{2}}},\underset{x_2}{\underset{\parallel}{3}},\underset{x_3}{\underset{\parallel}{\tfrac{7}{2}}},\underset{x_4}{\underset{\parallel}{4}},\underset{x_5}{\underset{\parallel}{\tfrac{9}{2}}},\underset{x_6}{\underset{\parallel}{5}}\Big\}$$

$$L(f, P_6) = \sum_{i=1}^{6} f(x_i^*)\,\Delta x_i$$

$$= \frac{1}{2}\left(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*) + f(x_6^*) \right)$$

$$= \frac{1}{2}\left(f(2) + \underbrace{f\left(\tfrac{5}{2}\right) + f\left(\tfrac{7}{2}\right)} + f(4) + f\left(\tfrac{9}{2}\right) + f(5) \right)$$

$$= \frac{35}{4}$$

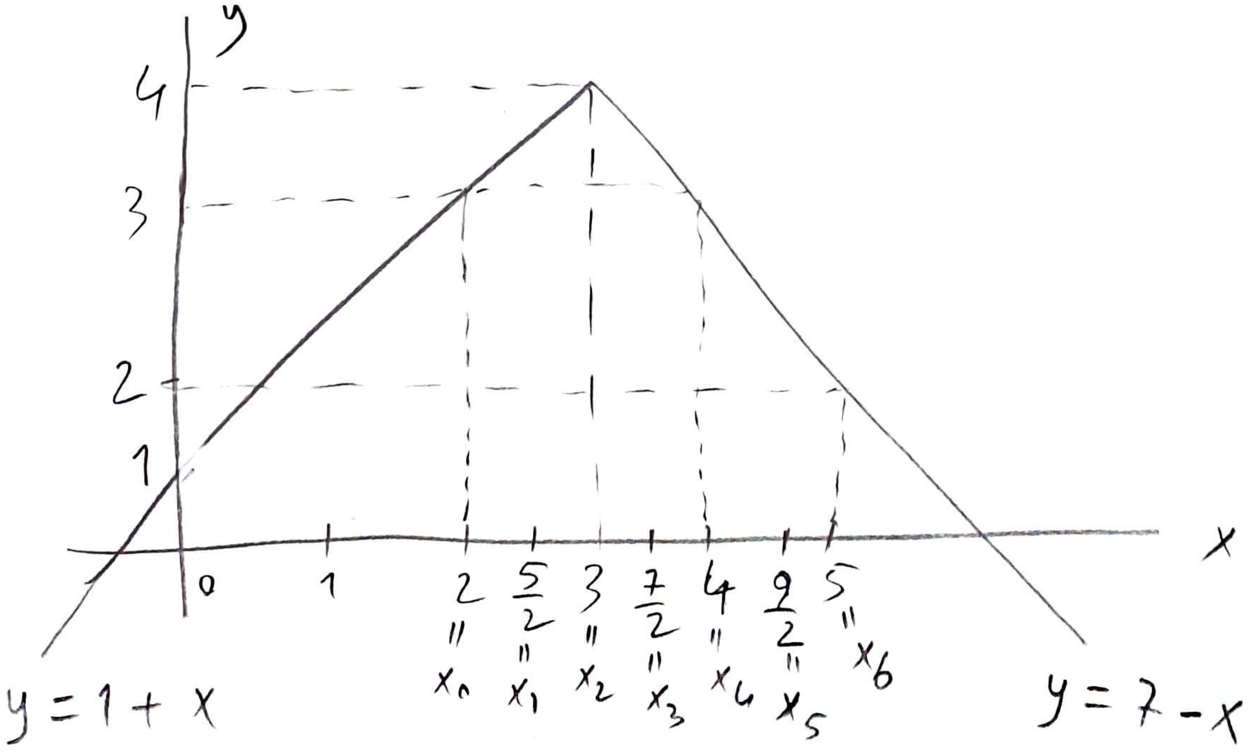
since f is increasing on $[2,3]$
f is decreasing on $[3,5]$

$$U(f, P_6) = \sum_{i=1}^{6} f(x_i^*)\,\Delta x_i = \frac{1}{2}\left(f(x_1^*) + f(x_2^*) + \dots + f(x_6^*) \right)$$

$$= \frac{1}{2}\left(f\left(\tfrac{5}{2}\right) + \underbrace{f(3) + f(3)} + f\left(\tfrac{7}{2}\right) + f(4) + f\left(\tfrac{9}{2}\right) \right)$$

$$= \frac{41}{4}$$

since f is increasing on $[2,3]$
f is decreasing on $[3,5]$



(c) <u>on $[2,3]$</u>: $f(x) = 1 + x$

$$\Delta x_i = \frac{3-2}{n} = \frac{1}{n}, \quad 1 \le i \le n$$

$$\Rightarrow P_n = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$$

where $x_i = 2 + i \cdot \Delta x_i = 2 + i \cdot \frac{1}{n}, \quad 0 \le i \le n$

so that $x_0 = 2$, $x_n = 3$

$$L(f, P_n) = \sum_{i=1}^{n} f(x_i^*) \underbrace{\Delta x_i}_{= \frac{1}{n}} = \frac{1}{n} \sum_{i=1}^{n} f(x_i^*)$$

$$= \frac{1}{n} \sum_{i=1}^{n} (1 + x_i^*) = \frac{1}{n} \sum_{i=1}^{n} (1 + x_{i-1})$$

$x_i^* = x_{i-1}$

(f is increasing on $[2,3]$)

$$= \frac{1}{n} \sum_{i=1}^{n} \left(1 + 2 + (i-1) \cdot \frac{1}{n}\right) = \frac{1}{n} \left(\sum_{i=1}^{n} 3 + (i-1) \cdot \frac{1}{n}\right)$$

$$= \frac{1}{n} \left(\sum_{i=1}^{n} 3 + \frac{1}{n} \sum_{i=1}^{n} (i-1)\right) = \frac{1}{n} \left(\sum_{i=1}^{n} 3 + \frac{1}{n} \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right)\right)$$

$$= \frac{1}{n} \left(3n + \frac{1}{n} \left(\frac{n(n+1)}{2} - n\right)\right)$$

$$= \frac{1}{n} \left(3n + \frac{n+1}{2} - 1\right) = \frac{7n-1}{2n}$$

$$U(f,P_n) = \sum_{i=1}^{n} f(x_i^*)\underbrace{\Delta x_i}_{= \frac{1}{n}} = \frac{1}{n}\sum_{i=1}^{n} f(x_i^*)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(1+x_i^*) = \frac{1}{n}\sum_{i=1}^{n}(1+x_i)$$

$x_i^* = x_i$

(since f is increasing on $[2,3]$)

$$= \frac{1}{n}\sum_{i=1}^{n}\left(1+2+i\cdot\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(3+i\cdot\frac{1}{n}\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}3 + \frac{1}{n}\sum_{i=1}^{n} i\right) = \frac{1}{n}\left(3n + \frac{1}{n}\cdot\frac{n(n+1)}{2}\right)$$

$$= \frac{1}{n}\left(3n + \frac{n+1}{2}\right) = \frac{7n+1}{2n}$$

<u>On $[3,5]$</u>: $f(x) = 7 - x$

$\Delta x_i = \frac{5-3}{n} = \frac{2}{n}$, $1 \le i \le n$

$\Rightarrow P_n = \{x_0, x_1, \dots, x_{n-1}, x_n\}$

where $x_i = 3 + i\cdot\Delta x_i = 3 + i\cdot\frac{2}{n}$, $0 \le i \le n$

so that $x_0 = 3$, $x_n = 5$

$$L(f,P_n) = \sum_{i=1}^{n} f(x_i^*) \underbrace{\Delta x_i}_{= \frac{2}{n}} = \frac{2}{n} \sum_{i=1}^{n} f(x_i^*)$$

$$= \frac{2}{n} \sum_{i=1}^{n} (7 - x_i^*) = \frac{2}{n} \sum_{i=1}^{n} (7 - x_i)$$

$x_i^* = x_i$

(since f is decreasing on $[3,5]$)

$$= \frac{2}{n} \sum_{i=1}^{n} \left(7 - 3 - i \cdot \frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^{n} \left(4 - i \cdot \frac{2}{n}\right)$$

$$= \frac{2}{n} \left(\sum_{i=1}^{n} 4 - \frac{2}{n} \sum_{i=1}^{n} i \right) = \frac{2}{n} \left(4n - \frac{2}{n} \cdot \frac{n(n+1)}{2} \right)$$

$$= \frac{2}{n} \left(4n - (n+1) \right) = \frac{6n-2}{n}$$

$$U(f,P_n) = \sum_{i=1}^{n} f(x_i^*) \underbrace{\Delta x_i}_{= \frac{2}{n}} = \frac{2}{n} \sum_{i=1}^{n} f(x_i^*)$$

$$= \frac{2}{n} \sum_{i=1}^{n} (7 - x_i^*) = \frac{2}{n} \sum_{i=1}^{n} (7 - x_{i-1})$$

$x_i^* = x_{i-1}$

(since f is decreasing on $[3,5]$)

$$= \frac{2}{n} \sum_{i=1}^{n} \left(7 - 3 - (i-1) \cdot \frac{2}{n}\right) = \frac{2}{n} \left(\sum_{i=1}^{n} 4 - \frac{2}{n} \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \right) \right)$$

$$= \frac{2}{n} \left(4n - \frac{2}{n} \left(\frac{n(n+1)}{2} - n \right) \right) = \frac{6n+2}{n}$$

(d) On $[2,3]$,

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{7n-1}{2n} = \lim_{n\to\infty} \frac{n(7-\frac{1}{n})}{2n} = \frac{7}{2}$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{7n+1}{2} = \lim_{n\to\infty} \frac{n(7+\frac{1}{n})}{2n} = \frac{7}{2}$$

On $[3,5]$,

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{6n-2}{n} = \lim_{n\to\infty} \frac{n(6-\frac{2}{n})}{n} = 6$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{6n+2}{n} = \lim_{n\to\infty} \frac{n(6+\frac{2}{n})}{n} = 6$$

So, on $[2,5]$,

$$\underset{(\text{on } [2,3])}{\lim_{n\to\infty} L(f,P_n)} + \underset{(\text{on } [3,5])}{\lim_{n\to\infty} L(f,P_n)} = \frac{7}{2} + 6 = \frac{19}{2}$$

$$\underset{(\text{on } [2,3])}{\lim_{n\to\infty} U(f,P_n)} + \underset{(\text{on } [3,5])}{\lim_{n\to\infty} U(f,P_n)} = \frac{7}{2} + 6 = \frac{19}{2}$$

In part (a)

$$\left.\begin{aligned} L(f,P_4) &= \frac{135}{16} \\ U(f,P_4) &= \frac{168}{16} \end{aligned}\right\} \Rightarrow U(f,P_4) - L(f,P_4) = \frac{33}{16}$$

In part (b),

$$\left.\begin{aligned} L(f,P_6) &= \frac{35}{4} \\ U(f,P_6) &= \frac{41}{4} \end{aligned}\right\} \Rightarrow U(f,P_6) - L(f,P_6) = \frac{6}{4} = \frac{24}{16}$$

So, as # of subintervals increases, then difference between upper and lower sums decreases, and accuracy increases.

(b) By properties of definite integral, evaluate the following definite integrals:

(a) $\int_{-2}^{2} \sinh(x) \ln\left(\sqrt{1+x^2}\right) + 4x^5 + 7 \, dx$

(b) $\int_{-1}^{2} \sqrt{4-x^2} + |x+3| + |x-3| \, dx$

(c) $\int_{-1}^{4} \operatorname{sgn}(x) \, dx$

Solution

(a) let $f(x) = \sinh(x) \ln\left(\sqrt{1+x^2}\right) + 4x^5$

$$f(-x) = \sinh(-x) \ln\left(\sqrt{1+(-x)^2}\right) + 4(-x)^5$$
$$= -\sinh(x) \ln\left(\sqrt{1+x^2}\right) - 4x^5$$
$$= -f(x)$$

So, f is odd. Then

$$\int_{-2}^{2} \underbrace{\sinh(x) \ln\left(\sqrt{1+x^2}\right) + 4x^5}_{\overset{\shortparallel}{f(x)}} + 7 \, dx$$

$$= \underbrace{\int_{-2}^{2} f(x) \, dx}_{\overset{\shortparallel}{0}} + \int_{-2}^{2} 7 \, dx = 28$$

(b) $\int_{-1}^{2} \sqrt{4-x^2} + |x+3| + |x-3| \, dx$

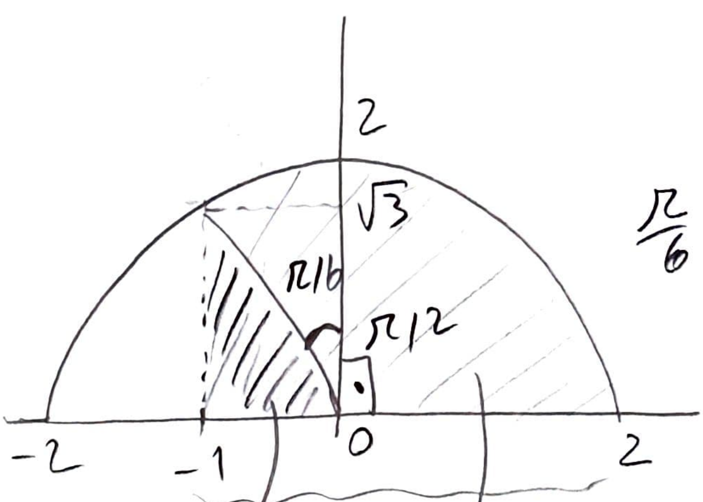
$= \int_{-1}^{2} \sqrt{4-x^2}\,dx + \int_{-1}^{2} |x+3|\,dx + \int_{-1}^{2} |x-3|\,dx$

$= \int_{-1}^{2} \sqrt{4-x^2}\,dx + \int_{-1}^{2} (x+3)\,dx + \int_{-1}^{2} (-x+3)\,dx$

$= \int_{-1}^{2} \sqrt{4-x^2}\,dx + \int_{-1}^{2} 6\,dx = \underbrace{\int_{-1}^{2} \sqrt{4-x^2}\,dx}_{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}} + 18$

$y = \sqrt{4-x^2}$

$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} + 18$



$\frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$

$\frac{\sqrt{3}}{2}$ $\quad \pi \cdot (2)^2 \cdot \frac{2\pi/3}{2\pi} = \frac{4\pi}{3}$

(c) $\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$\int_{-1}^{4} \operatorname{sgn}(x)\,dx = \int_{-1}^{0} \operatorname{sgn}(x)\,dx + \int_{1}^{4} \operatorname{sgn}(x)\,dx$

$= \int_{-1}^{0} (-1)\,dx + \int_{0}^{4} 1\,dx = -1 + 4 = 3$