Math119 2020-1 Recitation week 11

1) Evaluate the following integrals:

(a) $\displaystyle\int \frac{t^4-8}{t^3+2t^2}\,dt$ (b) $\displaystyle\int \frac{x^3+1}{x(x-1)^3}\,dx$

(c) $\displaystyle\int \frac{x^3+3x}{x^4+2x^2+1}\,dx$ (d) $\displaystyle\int \frac{2}{z^4-1}\,dz$

(e) $\displaystyle\int \frac{3x^2+2x+1}{(x+1)(x^2+x+1)}\,dx$ (f) $\displaystyle\int_0^1 \frac{x}{x^3+1}\,dx$

(g) $\displaystyle\int \frac{2-t^2}{t^3+3t^2+2t}\,dt$ (h) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$

Solution

(a)
$$\begin{array}{r|l} t^4-8 & t^3+2t^2 \\ \hline -t^4+2t^3 & t-2 \\ \hline -2t^3-8 & \\ -2t^3-4t^2 & \\ \hline 4t^2-8 & \end{array}$$

$$\Rightarrow \frac{t^4-8}{t^3+2t^2} = t-2+\frac{4t^2-8}{t^3+2t^2}$$

$$\Rightarrow \int \frac{t^4-8}{t^3+2t^2}\,dt = \int (t-2)\,dt + \int \frac{4t^2-8}{t^3+2t^2}\,dt$$

$$= \frac{t^2}{2} - 2t + C_1 + 4\int \frac{t^2-2}{t^3+2t^2}\,dt$$

$$\frac{t^2-2}{t^2(t+2)} = \frac{A}{t}+\frac{B}{t^2}+\frac{C}{t+2} = \frac{At(t+2)+B(t+2)+Ct^2}{t^2(t+2)}$$

$$= \frac{At^2+2At+Bt+2B+Ct^2}{t^2(t+2)}$$

$$\Rightarrow \frac{t^2-2}{t^2(t+2)} = \frac{(A+C)t^2+(2A+B)t+2B}{t^2(t+2)}$$

$$\Rightarrow \begin{aligned} A+C&=1 \\ 2A+B&=0 \\ 2B&=-2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} B&=-1 \\ A&=\frac{1}{2} \\ C&=\frac{1}{2} \end{aligned}$$

$$\Rightarrow \frac{t^2-2}{t^2(t+2)} = \frac{1}{2}\frac{1}{t} - \frac{1}{t^2} + \frac{1}{2}\frac{1}{t+2}$$

$$\Rightarrow \int \frac{t^2-2}{t^3+2t^2}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt - \int \frac{1}{t^2}\,dt + \frac{1}{2}\int \frac{1}{t+2}\,dt$$

$$= \frac{1}{2}\ln|t| - \left(-\frac{1}{t}\right) + \frac{1}{2}\ln|t+2| + C_2$$

Thus,

$$\int \frac{t^4-8}{t^3+2t^2}\,dt = \frac{t^2}{2} - 2t + 2\ln|t| + \frac{4}{t} + 2\ln|t+2| + C$$

(b) $\displaystyle\int \frac{x^3+1}{x(x-1)^3}\,dx$

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$= \frac{A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx}{x(x-1)^3}$$

$$\Rightarrow x^3+1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

$$x=0 \Rightarrow 1 = A(0-1)^3 + 0 + 0 + 0 \Rightarrow A=-1$$

$$x=1 \Rightarrow 2 = 0+0+0+D \Rightarrow D=2$$

$$\Rightarrow x^3+1 = -(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + 2x$$

Take derivative wrt x:

$$3x^2 = -3(x-1)^2 + B(x-1)^2 + 2Bx(x-1) + C(x-1) + Cx + 2$$

$$x=0 \Rightarrow 0 = -3 + B - C + 2$$

$$x=1 \Rightarrow 3 = C+2$$

$$\Rightarrow C=1, \quad B=2$$

$$\Rightarrow \frac{x^3+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}$$

$$\Rightarrow \int \frac{x^3+1}{x(x-1)^3}\,dx = -\int \frac{1}{x}\,dx + 2\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^3}\,dx$$

$$= -\ln|x| + 2\ln|x-1| + \int \frac{1}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^3}\,dx$$

$$\begin{pmatrix} a = x-1 \\ \Rightarrow da = dx \end{pmatrix} \quad \begin{pmatrix} b = x-1 \\ \Rightarrow db = dx \end{pmatrix}$$

$$= -\ln|x| + 2\ln|x-1| + \int \frac{1}{a^2}\,da + 2\int \frac{1}{b^3}\,db$$

$$= -\ln|x| + 2\ln|x-1| - \frac{1}{a} + 2\left(-\frac{1}{2b^2}\right) + C$$

$$= -\ln|x| + 2\ln|x-1| - \frac{1}{x-1} - \frac{1}{(x-1)^2} + C$$

ⓒ $\int \frac{x^3+3x}{x^4+2x^2+1}\,dx = \int \frac{x^3+3x}{(x^2+1)^2}\,dx$

$$\frac{x^3+3x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$= \frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)^2}$$

$$= \frac{Ax^3+Ax+Bx^2+B+Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$

$\Rightarrow A=1,\ B=0,\ A+C=3,\ B+D=0$

$\Rightarrow A=1,\ B=0,\ C=2,\ D=0$

$$\Rightarrow \int \frac{x^3+3x}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{2x}{(x^2+1)^2}\,dx$$

$\left(\begin{array}{l} a = x^2+1 \\ \Rightarrow da = 2x\,dx \\ \Rightarrow x\,dx = \frac{1}{2}\,da \end{array}\right)$ $\left(\begin{array}{l} b = x^2+1 \\ \Rightarrow db = 2x\,dx \end{array}\right)$

$$= \frac{1}{2}\int \frac{1}{a}\,da + \int \frac{1}{b^2}\,db = \frac{1}{2}\int \frac{1}{a}\,da + \int b^{-2}\,db$$

$$= \frac{1}{2}\ln|a| + \frac{b^{-1}}{-1} + C$$

$$= \frac{1}{2}\ln(x^2+1) - \frac{1}{x^2+1} + C$$

(d) $\int \frac{2}{z^4-1}\,dz = 2\int \frac{1}{(z^2-1)(z^2+1)}\,dz$

$= 2\int \frac{1}{(z-1)(z+1)(z^2+1)}\,dz$

$\frac{1}{(z-1)(z+1)(z^2+1)} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{Cz+D}{z^2+1}$

$= \frac{A(z+1)(z^2+1) + B(z-1)(z^2+1) + (Cz+D)(z-1)(z+1)}{(z-1)(z+1)(z^2+1)}$

$\Rightarrow 1 = A(z+1)(z^2+1) + B(z-1)(z^2+1) + (Cz+D)(z-1)(z+1)$

$z=1 \Rightarrow 1 = A(2)(2) + 0 + 0 \Rightarrow A = \frac{1}{4}$

$z=-1 \Rightarrow 1 = 0 + B(-2)(2) + 0 \Rightarrow B = -\frac{1}{4}$

$z=0 \Rightarrow 1 = A - B - D$
$= \frac{1}{4} + \frac{1}{4} - D \Rightarrow D = \frac{-1}{2}$

$\Rightarrow 1 = \frac{1}{4}(z+1)(z^2+1) - \frac{1}{4}(z-1)(z^2+1) + (Cz - \frac{1}{2})(z-1)(z+1)$

$z=2 \Rightarrow 1 = \frac{15}{4} - \frac{5}{4} + (2C - \frac{1}{2})(3)$
$= \frac{5}{2} + 6C - \frac{3}{2} = 6C + 1 \Rightarrow C = 0$

$\Rightarrow \int \frac{2}{z^4-1}\,dz = 2\int \frac{1}{(z-1)(z+1)(z^2+1)}\,dz$

$= 2\int\left(\frac{1}{4}\,\frac{1}{z-1} - \frac{1}{4}\,\frac{1}{z+1} - \frac{1}{2}\,\frac{1}{z^2+1}\right)dz$

$$= \frac{1}{2}\int \frac{1}{z-1}\,dz - \frac{1}{2}\int \frac{1}{z+1}\,dz - \int \frac{1}{z^2+1}\,dz$$

$$= \frac{1}{2}\ln|z-1| - \frac{1}{2}\ln|z+1| - \arctan(z) + C$$

(e) $\displaystyle\int \frac{3x^2+2x+1}{(x+1)(x^2+x+1)}\,dx$

$$\frac{3x^2+2x+1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1)+(Bx+C)(x+1)}{(x+1)(x^2+x+1)}$$

$$= \frac{Ax^2+Ax+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+x+1)}$$

$$= \frac{(A+B)x^2+(A+B+C)x+(A+C)}{(x+1)(x^2+x+1)}$$

$\Rightarrow A+B=3,\ A+B+C=2,\ A+C=1$

$\Rightarrow C=-1,\ A=2,\ B=1$

$$\Rightarrow \int \frac{3x^2+2x+1}{(x+1)(x^2+x+1)}\,dx = 2\int \frac{1}{x+1}\,dx + \int \frac{x-1}{x^2+x+1}\,dx$$

$$= 2\ln|x+1| + \boxed{\int \frac{x-1}{x^2+x+1}\,dx} \quad \left(\begin{array}{l} u = x^2+x+1 \\ \Rightarrow du = (2x+1)\,dx \end{array}\right)$$

Write $x-1 = \frac{1}{2}(2x+1) - \frac{3}{2}$

So, $\int \frac{x-1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$

$= \frac{1}{2}\int \frac{1}{u}du - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$

$= \frac{1}{2}\ln|u| - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$

$= \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$

Consider $\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$

$= \int \frac{1}{\frac{3}{4}\left(1+\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2\right)}dx$

$= \frac{4}{3}\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}dx$

$$\left(\begin{aligned} u &= \frac{2x+1}{\sqrt{3}} \\ \Rightarrow du &= \frac{2}{\sqrt{3}}dx \\ \Rightarrow dx &= \frac{\sqrt{3}}{2}du \end{aligned}\right)$$

$= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{1+u^2}du$

$= \frac{2}{\sqrt{3}}\arctan(u) + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$

Thus, $\int \frac{3x^2+2x+1}{(x+1)(x^2+x+1)}dx$

$= 2\ln|x+1| + \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$

(f) $\int_0^1 \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{(x+1)(x^2-x+1)}\,dx$

$$\frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$= \frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{(x+1)(x^2-x+1)}$$

$$= \frac{(A+B)x^2+(-A+B+C)x+(A+C)}{(x+1)(x^2-x+1)}$$

$\Rightarrow A+B=0,\ A+C=0,\ -A+B+C=1$

$\Rightarrow 2A+B+C=0,\ -A+B+C=1$

$\Rightarrow A=\frac{-1}{3},\ B=+\frac{1}{3},\ C=+\frac{1}{3}$

$$\Rightarrow \frac{x}{(x+1)(x^2-x+1)} = \frac{-1}{3}\,\frac{1}{x+1} + \frac{1}{3}\,\frac{x+1}{x^2-x+1}$$

$$\Rightarrow \int_0^1 \frac{x}{x^3+1}\,dx = -\frac{1}{3}\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{3}\int_0^1 \frac{x+1}{x^2-x+1}\,dx$$

$$= \underbrace{-\frac{1}{3}\ln|x+1|\Big|_0^1}_{=\,-\frac{1}{3}\ln 2} + \frac{1}{3}\boxed{\int_0^1 \frac{x+1}{x^2-x+1}\,dx}$$

$u = x^2-x+1$
$\Rightarrow du = (2x-1)\,dx$

write $x+1 = \frac{1}{2}(2x-1) + \frac{3}{2}$

$$\int_0^1 \frac{x+1}{x^2-x+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x-1}{x^2-x+1}\,dx + \frac{3}{2}\int_0^1 \frac{1}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\int_1^1 \frac{1}{u}\,du + \frac{3}{2}\int_0^1 \frac{1}{x^2-x+1}\,dx$$

$$= \frac{3}{2}\int_0^1 \frac{1}{x^2-x+1}\,dx$$

$$\int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int_0^1 \frac{1}{\frac{3}{4}\left(1+\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2\right)}\,dx$$

$$= \frac{4}{3}\int_0^1 \frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\,dx \qquad \left(\begin{aligned} &u = \frac{2x-1}{\sqrt{3}} \\ &\Rightarrow du = \frac{2}{\sqrt{3}}\,dx \\ &\Rightarrow dx = \frac{\sqrt{3}}{2}\,du \end{aligned} \right)$$

$$= \frac{4}{3}\frac{\sqrt{3}}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{1}{1+u^2}\,du$$

$$= \frac{2}{\sqrt{3}} \arctan(u)\Big|_{-1/\sqrt{3}}^{1/\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left(\underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{= \pi/6} - \underbrace{\arctan\left(\frac{-1}{\sqrt{3}}\right)}_{= -\pi/6}\right)$$

$$= \frac{2}{\sqrt{3}}\frac{2\pi}{6} = \frac{2\pi}{3\sqrt{3}}$$

$$\Rightarrow \frac{3}{2}\int_0^1 \frac{1}{x^2-x+1}\,dx = \frac{\cancel{3}}{\cancel{2}}\frac{\cancel{2}\pi}{\cancel{3}\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

So, $\int_0^1 \frac{x}{x^3+1}\,dx = -\frac{1}{3}\ln 2 + \frac{1}{3}\frac{\pi}{\sqrt{3}}$

⑨ $$\int \frac{2-t^2}{t^3+3t^2+2t}\,dt = \int \frac{2-t^2}{t(t+1)(t+2)}\,dt$$

$$\frac{2-t^2}{t(t+1)(t+2)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t+2}$$

$$= \frac{A(t+1)(t+2) + Bt(t+2) + Ct(t+1)}{t(t+1)(t+2)}$$

$$= \frac{A(t^2+3t+2) + B(t^2+2t) + C(t^2+t)}{t(t+1)(t+2)}$$

$$= \frac{(A+B+C)t^2 + (3A+2B+C)t + 2A}{t(t+1)(t+2)}$$

$$\Rightarrow \begin{array}{l} A+B+C=-1 \\ 3A+2B+C=0 \\ 2A=2 \Rightarrow A=1 \end{array} \quad \Rightarrow \quad \begin{array}{l} B+C=-2 \\ 2B+C=-3 \end{array}$$

$$\Rightarrow A=1,\ B=-1,\ C=-1$$

$$\Rightarrow \int \frac{2-t^2}{t^3+3t^2+2t}\,dt = \int \left(\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right) dt$$

$$= \ln|t| - \ln|t+1| - \ln|t+2| + C$$

ⓗ $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin(x) + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1 - x^2 \longrightarrow du = -2x\,dx$

$\Rightarrow x\,dx = -\frac{1}{2}\,du$

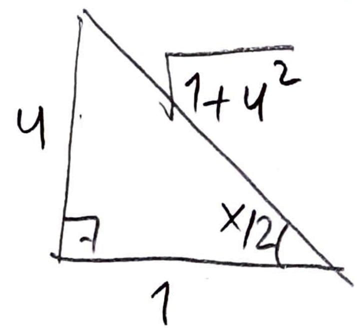
$\Rightarrow \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2} \int \frac{1}{\sqrt{u}}\,du$

$= -\frac{1}{2} \frac{u^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

$\rightarrow \int \sqrt{\frac{1+x}{1-x}}\,dx = \arcsin(x) - \sqrt{1-x^2} + C$

② Use $u = \tan\frac{x}{2}$ substitution to evaluate $\int \frac{dx}{3\sin x + 4\cos x}$.

Solution



Let $u = \tan\frac{x}{2}$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$$

$$= 2\left(\frac{1}{\sqrt{1+u^2}}\right)^2 - 1 = \frac{2}{1+u^2} - 1$$

$$= \frac{2-1-u^2}{1+u^2} = \frac{1-u^2}{1+u^2}$$

$$\Rightarrow \frac{1}{3\sin x + 4\cos x} = \frac{1}{\frac{6u}{1+u^2} + \frac{4-4u^2}{1+u^2}}$$

$$= \frac{1+u^2}{4+6u-4u^2}$$

Also, $u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

$$\Rightarrow du = \frac{1}{2}\frac{1}{\cos^2\left(\frac{x}{2}\right)}dx = \frac{1}{2}\frac{1}{\frac{1}{1+u^2}}dx$$

$$\Rightarrow dx = \frac{2}{1+u^2}du$$

$$\Rightarrow \int \frac{1}{3\sin x+4\cos x}\,dx$$

$$= \int \frac{1+u^2}{4+6u-4u^2}\,\frac{2}{1+u^2}\,du = -\int \frac{1}{2u^2-3u-2}\,du$$

$$= \int \frac{-1}{(u-2)(2u+1)}\,du$$

$$\frac{-1}{(u-2)(2u+1)} = \frac{A}{u-2} + \frac{B}{2u+1}$$

$$= \frac{A(2u+1)+B(u-2)}{(u-2)(2u+1)} = \frac{(2A+B)u+(A-2B)}{(u-2)(2u+1)}$$

$$\Rightarrow \begin{matrix} 2A+B=0 \\ A-2B=-1 \end{matrix} \Rightarrow \begin{matrix} 4A+2B=0 \\ A-2B=-1 \end{matrix} \Rightarrow \begin{matrix} A=-\frac{1}{5} \\ B=\frac{2}{5} \end{matrix}$$

$$\Rightarrow \int \frac{1}{3\sin x+4\cos x}\,dx = -\frac{1}{5}\int \frac{1}{u-2}\,du + \frac{2}{5}\int \frac{1}{2u+1}\,du$$

$$= -\frac{1}{5}\ln|u-2| + \frac{2}{5}\,\frac{\ln|2u+1|}{2} + C$$

$$= -\frac{1}{5}\ln\left|\tan\left(\frac{x}{2}\right)-2\right| + \frac{1}{5}\ln\left|2\tan\left(\frac{x}{2}\right)+1\right| + C$$

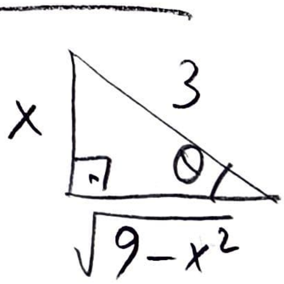
③ Using inverse trigonometric substitution, evaluate the following integrals:

ⓐ $\displaystyle\int \frac{\sqrt{9-x^2}}{x}\,dx$ ⓑ $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}$

ⓒ $\displaystyle\int \frac{dx}{(1+x^2)^2}$

<u>Solution</u>

ⓐ



$\left(\theta = \arcsin \frac{x}{3},\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\sin\theta = \frac{x}{3} \Rightarrow \cos\theta\, d\theta = \frac{1}{3}\,dx$

$\Rightarrow dx = 3\cos\theta\, d\theta$

$x = 3\sin\theta$

$$\Rightarrow \sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3|\cos\theta|$$
$$= 3\cos\theta$$

$$\Rightarrow \int \frac{\sqrt{9-x^2}}{x}\,dx = \int \frac{3\cos\theta}{3\sin\theta}\, 3\cos\theta\, d\theta$$

$$= 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= 3\int \frac{1}{\sin\theta}\,d\theta - 3\int \sin\theta\, d\theta$$

$$= 3\int \frac{1}{\sin\theta}\,d\theta + 3\cos\theta + C$$

$$= 3\int \frac{1}{\sin\theta}\,d\theta + \sqrt{9-x^2} + C$$

$$= 3\int \frac{\csc(\theta)\,(\csc\theta + \cot\theta)}{\csc\theta + \cot\theta}\,d\theta + \sqrt{9-x^2} + C$$

$u = \csc\theta + \cot\theta$

$\Rightarrow du = (-\csc^2\theta - \cot\theta\csc\theta)\, d\theta$

$\Rightarrow \int \frac{1}{\sin\theta}\, d\theta = \int \frac{\csc\theta(\csc\theta + \cot\theta)}{\csc\theta + \cot\theta}\, d\theta$

$= -\int \frac{1}{u}\, du = -\ln|u| + \tilde{C}$

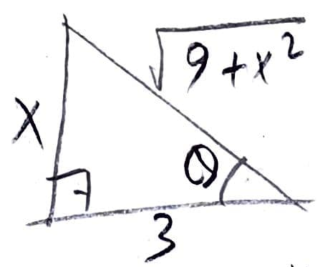
$= -\ln|\csc\theta + \cot\theta| + \tilde{C}$

$\Rightarrow \int \frac{\sqrt{9-x^2}}{x}\, dx$

$= -3\ln|\csc\theta + \cot\theta| + \sqrt{9-x^2} + C$

$= -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + \sqrt{9-x^2} + C$

(b) $\int \frac{dx}{(9+x^2)^{3/2}}$



let $\tan\theta = \frac{x}{3}$, $\left(\theta = \arctan\frac{x}{3},\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\Rightarrow \sec^2\theta\, d\theta = \frac{1}{3}\, dx \Rightarrow dx = 3\sec^2\theta\, d\theta$

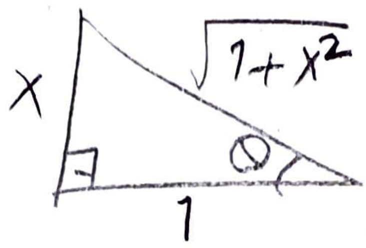
$9 + x^2 = 9 + 9\tan^2\theta = 9(1+\tan^2\theta) = 9\sec^2\theta$

$\Rightarrow \int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{9^{3/2}(\sec^2\theta)^{3/2}}$

$= \int \frac{3\sec^2\theta\, d\theta}{27\sec^3\theta} = \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta$

$$= \frac{1}{9}\sin\theta + C = \frac{1}{9}\frac{x}{\sqrt{9+x^2}} + C$$

ⓒ $\displaystyle\int \frac{dx}{(1+x^2)^2}$



let $\tan\theta = x$. ($\theta = \arctan x$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

$$\Rightarrow \sec^2\theta \, d\theta = dx$$

$$1+x^2 = 1+\tan^2\theta = \sec^2\theta$$

$$\Rightarrow \int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \int \cos^2\theta \, d\theta$$

$$= \int \frac{1+\cos(2\theta)}{2} d\theta = \int \frac{1}{2} d\theta + \frac{1}{2}\int \cos(2\theta) d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

$$= \frac{\arctan(x)}{2} + \frac{2\sin\theta\cos\theta}{4} + C$$

$$= \frac{\arctan(x)}{2} + \frac{1}{2}\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{\arctan(x)}{2} + \frac{x}{2+2x^2} + C$$