Math 119 2020-1 Recitation Problems Week 10

① Let $P = \{0,1,2,3\}$ be the partition of $[0,3]$ into three subintervals of equivalent length. Sketch roughly the graph of a nonnegative continuous function $f: [0,3] \longrightarrow \mathbb{R}$ such that the upper Riemann sum $U(f,P) = \frac{1}{2} + \frac{1}{2} + \frac{3}{2}$ and the lower Riemann sum $L(f,P) = 0 + 0 + \frac{1}{2}$.

Solution

We have three subintervals; $[0,1]$, $[1,2]$, $[2,3]$.

On $[0,1]$, lower Riemann sum is 0

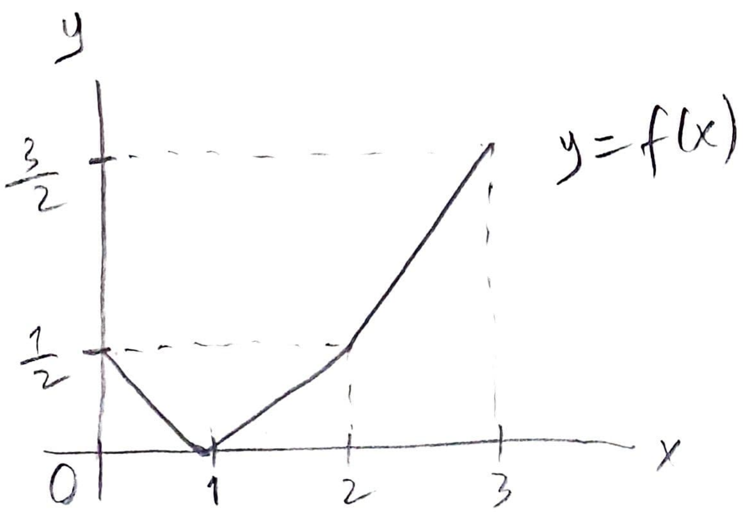
upper —— " —— $\frac{1}{2}$

On $[1,2]$, lower —— " —— 0

upper —— " —— $\frac{1}{2}$

On $[2,3]$, lower —— " —— $\frac{1}{2}$

upper —— " —— $\frac{3}{2}$



(2) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[1-\left(-1+\frac{2i}{n}\right)^3\right]$

as a definite integral and evaluate it.

Solution

Let $f(x) = 1 - x^3 \Rightarrow f(x_i) = 1 - x_i^3$ where

$x_i = -1 + i \cdot \frac{2}{n} \Rightarrow \Delta x_i = \frac{2}{n}$

$x_0 = -1 + 0 \cdot \frac{2}{n} = -1$

$x_n = -1 + n \cdot \frac{2}{n} = -1 + 2 = 1$

So,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[1-\left(-1+\frac{2i}{n}\right)^3\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x_i f(x_i)$$

$$= \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (1-x^3)\,dx = \int_{-1}^{1} 1\,dx - \underbrace{\int_{-1}^{1} x^3\,dx}_{= 0}$$

$= 2$

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(3) Find the average value of $f(x) = \frac{1}{x}$ on $[\frac{1}{3}, 3]$.

Solution: Average value of $f(x)$ on $[a, b]$ is

$$\text{Avg}(f) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\text{So, } \text{Avg}(f) = \bar{f} = \frac{1}{3-\frac{1}{3}}\int_{\frac{1}{3}}^{3} \frac{1}{x}\,dx = \frac{3}{8} \ln x \Big|_{\frac{1}{3}}^{3}$$

$$= \frac{3}{8}\left(\ln 3 - \ln \frac{1}{3}\right) = \frac{3}{8}\left(\ln 3 - \ln 3^{-1}\right)$$

$$= \frac{3}{8} 2 \ln 3 = \frac{3}{4} \ln 3$$

(4) Evaluate

$$\lim_{x\to 0} \frac{\int_{-2}^{0} \sin(t^2)\,dt - \int_{-2}^{x^2} \sin(t^2)\,dt}{\int_{\cos x}^{1} e^{\sqrt{t}-1}-1\,dt} = A$$

Solution

Recall (FTC): $\frac{d}{dx}\left(\int_{f(x)}^{g(x)} h(t)\,dt = h(g(x)).g'(x) - h(f(x)).f'(x)\right.$

$$\frac{d}{dx}\left(\int_{-2}^{0} \sin(t^2)\,dt - \int_{-2}^{x^2} \sin(t^2)\,dt \right)$$

$$= \underbrace{\cancel{\frac{d}{dx} \int_{-2}^{0} \sin(t^2)\,dt}}_{0} - \frac{d}{dx} \int_{-2}^{x^2} \sin(t^2)\,dt$$

$$= -\, 2x \cdot \sin((x^2)^2) = -2x \cdot \sin(x^4)$$

$$\frac{d}{dx} \int_{\cos x}^{1} e^{\sqrt{t}-1}-1\,dt = -(e^{\sqrt{\cos x}-1}-1)(-\sin x)$$

$$\text{So, } A = \lim_{x\to 0} \frac{0 - 2x\sin(x^4) \quad . x^4}{-(e^{\sqrt{\cos x}-1}-1)(-\sin x) . x^4}$$

$$= +2 \underbrace{\lim_{x\to 0} \frac{x^4}{e^{\sqrt{\cos x}-1}-1}}_{=0} \; \underbrace{\lim_{x\to 0} \frac{x}{\sin x}}_{=1} \; \underbrace{\lim_{x\to 0} \frac{\sin(x^4)}{x^4}}_{=1} = 0$$

since each limit exists

where

$$\lim_{x \to 0} \frac{x^4}{e^{\sqrt{\cos x} - 1} - 1} \overset{\left[\frac{0}{0}\right]}{\underset{\text{L'Hopital}}{=}} \lim_{x \to 0} \frac{4x^3}{e^{\sqrt{\cos x} - 1} \cdot \left(\frac{-\sin x}{2\sqrt{\cos x}} \right)}$$

$$= \lim_{x \to 0} \frac{4x \cdot x^2}{e^{\sqrt{\cos x} - 1} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)}$$

$$= \lim_{x \to 0} \left(\underbrace{\frac{-4}{e^{\sqrt{\cos x} - 1}}}_{\to -4} \cdot \underbrace{\frac{x}{\sin x}}_{\to 1} \cdot \underbrace{2\sqrt{\cos x}}_{\to 2} \cdot \underbrace{x^2}_{\to 0} \right)$$

$$= 0$$

⑤ Evaluate the following define or indefinite integrals!

ⓐ $\int_0^2 x^5 \sqrt{x^2+3}\, dx$

Solution

$u = x^2+3 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2} du = x\,dx$

$u = x^2+3 \Rightarrow x^2 = u-3 \Rightarrow x^4 = (u-3)^2$

$x = 0 \Rightarrow u = 3$

$x = 2 \Rightarrow u = 7$

$$\int_0^2 x^5 \sqrt{x^2+3}\, dx = \int_0^2 x^4 \sqrt{x^2+3}\; x\, dx$$

$$= \int_3^7 (u-3)^2 \sqrt{u}\; \frac{1}{2} du = \frac{1}{2} \int_3^7 (u^2-6u+9)\sqrt{u}\, du$$

$$= \frac{1}{2} \int_3^7 (u^{5/2} - 6u^{3/2} + 9u^{1/2})\, du$$

$$= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - 6\frac{u^{5/2}}{5/2} + 9\frac{u^{3/2}}{3/2} \right) \Bigg|_3^7$$

$$= \left(\frac{1}{7} 7^{7/2} - \frac{6}{5} 7^{5/2} + \frac{1}{3} 7^{3/2}\right) - \left(\frac{1}{7} 3^{7/2} - \frac{6}{5} 3^{5/2} + \frac{1}{3} 3^{3/2}\right)$$

(b) $$\int \frac{2x+3}{4x^2+4x+3}\,dx = \int \frac{2x+1+2}{4x^2+4x+3}\,dx$$

$$= \underbrace{\int \frac{2x+1}{4x^2+4x+3}\,dx}_{A} + \underbrace{\int \frac{2}{4x^2+4x+3}\,dx}_{B}$$

$$A = \int \frac{2x+1}{4x^2+4x+3}\,dx \qquad \left(\begin{array}{l} u = 4x^2+4x+3 \\ \Rightarrow du = (8x+4)\,dx \end{array} \right)$$

$$= \frac{1}{4}\int \frac{8x+4}{4x^2+4x+3}\,dx$$

$$= \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C_1$$

$$= \frac{1}{4}\ln|4x^2+4x+3| + C_1$$

$$B = \int \frac{2}{4x^2+4x+3}\,dx \qquad \left(\begin{array}{l} u = \frac{2x+1}{\sqrt{2}} \\ \Rightarrow du = \sqrt{2}\,dx \\ \Rightarrow dx = \frac{1}{\sqrt{2}}\,du \end{array} \right)$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{\sqrt{2}}\arctan(u) + C_2 = \frac{1}{\sqrt{2}}\arctan\left(\frac{2x+1}{\sqrt{2}}\right) + C_2$$

Hence, $\displaystyle\int \frac{2x+3}{4x^2+4x+3}\,dx$

(c) $\int \frac{\sin(2x)}{1+\sin^2(x)}\,dx$

Solution

$$\int \frac{\sin(2x)}{1+\sin^2(x)}\,dx = \int \frac{2\sin(x)\cos(x)}{1+\sin^2(x)}\,dx$$

$$= \int \frac{2u\,du}{1+u^2} \qquad \left(\begin{array}{l} u=\sin x \\ \Rightarrow du = \cos x\,dx \end{array}\right)$$

$(v = 1+u^2 \Rightarrow dv = 2u\,du)$

$$= \int \frac{dv}{v} = \ln|v| + C = \ln|1+u^2| + C$$

$$= \ln|1+\sin^2 x| + C$$

(d) $\int \frac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}\,dx \qquad \left(\begin{array}{l} u=\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \\ \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx \end{array}\right)$

$$= 2\int \sec u \tan u\,du$$

$$= 2\sec u + C = 2\sec(\sqrt{x}) + C$$

(e) $\int \sin^3 x \sec^5 x\,dx = \int \sin^3 x \frac{1}{\cos^5 x}\,dx$

$$= \int \frac{1}{\cos^2 x}\frac{\sin^3 x}{\cos^3 x}\,dx = \int \sec^2 x \tan^3 x\,dx$$

$(u = \tan x \Rightarrow du = \sec^2 x\,dx)$

$$= \int u^3\,du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$$

(f) $\int_1^e \sqrt[3]{x}\ \ln(x)\,dx$ $\left(\begin{array}{l} u=\ln x \Rightarrow du=\frac{1}{x}dx \\ dv=\sqrt[3]{x}\,dx=x^{1/3}dx \\ \qquad \Rightarrow v=\frac{3}{4}x^{4/3} \end{array}\right)$

$= \ln x\ \frac{3}{4}x^{4/3}\Big|_1^e - \int_1^e \frac{3}{4}x^{4/3}\frac{1}{x}dx$

integration by parts

$= \frac{3}{4}e^{4/3} - \frac{3}{4}\int_1^e x^{1/3}dx = \frac{3}{4}e^{4/3} - \frac{9}{16}x^{4/3}\Big|_1^e$

$= \frac{3}{4}e^{4/3} - \frac{9}{16}e^{4/3} + \frac{9}{16} = \frac{3}{16}e^{4/3} + \frac{9}{16}$

(g) $\int \frac{\cos(\ln x)}{x^3}dx$ $\left(\begin{array}{l} u=\ln x \Rightarrow du=\frac{1}{x}dx \\ \Downarrow \\ x=e^u \end{array}\right)$

$= \int \frac{\cos(\ln x)}{x^2}\frac{dx}{x} = \int \frac{\cos u}{x^2}du = \int \frac{\cos u}{e^{2u}}du$

$= \int \underbrace{\cos u}_{a}\cdot\underbrace{e^{-2u}du}_{db}$ $\left(\begin{array}{l} a=\cos u \Rightarrow da=-\sin u\,du \\ db=e^{-2u}du \Rightarrow b=\frac{e^{-2u}}{-2} \end{array}\right)$

$= \cos u\cdot\frac{e^{-2u}}{-2} - \int \frac{e^{-2u}}{-2}\cdot(-\sin u)\,du$

$= \frac{-1}{2}\cos u\cdot e^{-2u} - \frac{1}{2}\int \sin u\cdot e^{-2u}du$

$= -\frac{1}{2}\cos(\ln x)\cdot\underbrace{e^{-2\ln x}}_{x^{-2}} - \frac{1}{2}\int \sin u\cdot e^{-2u}du$

$$= -\frac{x^{-2}}{2}\cos(\ln x) - \frac{1}{2}\underbrace{\int \sin u \cdot e^{-2u}\, du}$$

$$\left(\begin{array}{l} a = \sin u \Rightarrow da = \cos u\, du \\ db = e^{-2u}\, du \Rightarrow b = \frac{e^{-2u}}{-2} \end{array} \right)$$

$$= -\frac{x^{-2}}{2}\cos(\ln x) - \frac{1}{2}\left[\frac{\sin u \cdot e^{-2u}}{-2} - \int \frac{\cos u \cdot e^{-2u}}{-2}\, du\right]$$

Thus,

$$\underbrace{\int \cos u \cdot e^{-2u}\, du} = \frac{-x^{-2}\cos(\ln x)}{2} + \frac{1}{4}\sin u \cdot e^{-2u} - \frac{1}{4}\underbrace{\int \cos u \cdot e^{-2u}\, du}$$

$$\Rightarrow \frac{5}{4}\int \cos u \cdot e^{-2u}\, du = \frac{-x^{-2}\cos(\ln x)}{2} + \frac{1}{4}\sin u \cdot e^{-2u}$$

$$\Rightarrow \int \cos u \cdot e^{-2u}\, du = \frac{4}{5}\left(\frac{-x^{-2}\cos(\ln x)}{2} + \frac{1}{4}\sin u \cdot e^{-2u}\right)$$

$$= \frac{4}{5}\left(\frac{-x^{-2}\cos(\ln x)}{2} + \frac{1}{4}\sin(\ln x) \cdot \underbrace{e^{-2\ln x}}_{\substack{\parallel \\ x^{-2}}}\right)$$

$$= \frac{-2x^{-2}\cos(\ln x)}{5} + \frac{1}{5}x^{-2}\sin(\ln x)$$

Hence,

$$\int \frac{\cos(\ln x)}{x^3}\, dx = -\frac{2}{5}x^{-2}\cos(\ln x) + \frac{1}{5}x^{-2}\sin(\ln x) + C$$

(h) $\int \frac{t}{\sqrt{9-t^4}}\,dt = \int \frac{t}{\sqrt{9\left(1-\frac{t^4}{9}\right)}}\,dt$

$= \frac{1}{3}\int \frac{t}{\sqrt{1-\left(\frac{t^2}{3}\right)^2}}\,dt$ $\left(\begin{array}{l} u = \frac{t^2}{3} \Rightarrow du = \frac{2}{3}t\,dt \\ \Rightarrow t\,dt = \frac{3}{2}\,du \end{array}\right)$

$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,\frac{3}{2}\,du = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin\left(\frac{t^2}{3}\right) + C$

(i) $\int x^2 \arcsin(x)\,dx$ $\left(\begin{array}{l} u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx \\ dv = x^2\,dx \Rightarrow v = \frac{x^3}{3} \end{array}\right)$

integration by parts

$\stackrel{\text{integration by parts}}{=} \frac{x^3}{3}\arcsin x - \int \frac{x^3}{3}\,\frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\underbrace{\int \frac{x^2}{\sqrt{1-x^2}}\,x\,dx}$

$\left(\begin{array}{l} u = 1-x^2 \Rightarrow du = -2x\,dx \\ \Rightarrow x\,dx = -\frac{1}{2}\,du \\ \text{and } x^2 = 1-u \end{array}\right)$

So,

$\int \frac{x^2}{\sqrt{1-x^2}}\,x\,dx = \int \frac{1-u}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du$

$= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) = \frac{1}{2}\left(\frac{(1-x^2)^{3/2}}{3/2} - \frac{(1-x^2)^{1/2}}{1/2}\right)$$

So, $\int x^2 \arcsin(x)\,dx$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\left(\frac{1}{2}\left(\frac{2}{3}(1-x^2)^{3/2} - 2(1-x^2)^{1/2}\right)\right) + C$$

j) $\int_2^{\sqrt{5}} x^2 e^{3x}\,dx$ $\quad\left(\begin{array}{l} u = x^2 \Rightarrow du = 2x\,dx \\ dv = e^{3x}\,dx \Rightarrow v = \frac{e^{3x}}{3} \end{array}\right)$

integration by parts $= x^2 \frac{e^{3x}}{3}\Big|_2^{\sqrt{5}} - \int_2^{\sqrt{5}} 2x\,\frac{e^{3x}}{3}\,dx$

$$= \frac{5}{3}e^{3\sqrt{5}} - \frac{4}{3}e^6 - \frac{2}{3}\int_2^{\sqrt{5}} 2x\,\frac{e^{3x}}{3}\,dx$$

$$= \frac{5}{3}e^{3\sqrt{5}} - \frac{4}{3}e^6 - \frac{2}{3}\int_2^{\sqrt{5}} x e^{3x}\,dx$$

$$\left(\begin{array}{l} u = x \Rightarrow dv = dx \\ dv = e^{3x}\,dx \Rightarrow v = \frac{e^{3x}}{3} \end{array}\right)$$

$$= \frac{5}{3}e^{3\sqrt{5}} - \frac{4}{3}e^6 - \frac{2}{3}\left(\frac{x e^{3x}}{3}\Big|_2^{\sqrt{5}} - \int_2^{\sqrt{5}} \frac{e^{3x}}{3}\,dx\right)$$

$$= \frac{5}{3}e^{3\sqrt{5}} - \frac{4}{3}e^6 - \frac{2}{3}\left[\frac{\sqrt{5}e^{3\sqrt{5}}}{3} - \frac{2e^6}{3} - \frac{1}{3}\,\frac{e^{3x}}{3}\Big|_2^{\sqrt{5}}\right]$$

$$= \frac{5}{3} e^{3\sqrt{5}} - \frac{4}{3} e^{6} - \frac{2}{9}\sqrt{5}\, e^{3\sqrt{5}} + \frac{4}{9} e^{6}$$

$$+ \frac{2}{9}\left(\frac{e^{3\sqrt{5}}}{3} - \frac{e^{6}}{3} \right)$$

$$= \frac{6\sqrt{5} - 47}{27} e^{3\sqrt{5}} - \frac{26}{27} e^{6}$$