$$f(x) = \begin{cases} \sin x, & x \geq 0 \\ x^2 + x, & x < 0 \end{cases}$$

Show that f is differentiable at $x=0$. (using defn of derivative)

Consider $\displaystyle \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$

since $f(0) = \sin(0) = 0$

- $\displaystyle \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$, $\left[f(x) = \sin x \text{ if } x \geq 0 \right\}$

- $\displaystyle \lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{x^2 + x}{x} = 1$, $\left[f(x) = x^2 + x \text{ if } x < 0 \right]$

Since $\displaystyle \lim_{x \to 0^-} \frac{f(x)}{x} = 1 = \lim_{x \to 0^+} \frac{f(x)}{x}$, $\displaystyle \lim_{x \to 0} \frac{f(x)}{x}$ exist

and $\displaystyle f'(0) = \lim_{x \to 0} \frac{f(x)}{x}$

Hence, f is differentiable at $x = 0$.