

Let f be a differentiable function with continuous derivative, and let $f(0) = 0, f(1) = 3, f'(0) = f'(1) = 1$.

$$\text{Find } \lim_{x \rightarrow 0} \frac{f(2x+1) - 3}{f(4x)}.$$

Since f is continuous and both $2x+1$
and $4x$ are continuous, composition of these
functions, i.e. $f(2x+1)$ and $f(4x)$ will be
continuous.

$$\text{That is } \lim_{x \rightarrow 0} f(2x+1) = f\left(\lim_{x \rightarrow 0} 2x+1\right) = f(1) = 3$$

$$\& \lim_{x \rightarrow 0} f(4x) = f\left(\lim_{x \rightarrow 0} 4x\right) = f(0) = 0$$

This means $\lim_{x \rightarrow 0} \frac{f(2x+1) - 3}{f(4x)}$ is indeterminate

type of $\left[\frac{0}{0} \right]$.

$$\text{If we consider } \lim_{x \rightarrow 0} \frac{\left[f(2x+1) - 3 \right]'}{\left[f(4x) \right]'} = \lim_{x \rightarrow 0} \frac{f'(2x+1) \cdot 2}{f'(4x) \cdot 4}$$

by the
Chain Rule.

But it is given that f' is also continuous, therefore;

$$\lim_{x \rightarrow 0} \frac{f'(2x+1) \cdot 2}{f'(4x) \cdot 4} = \frac{\underset{x=1}{f'(1) \cdot 2}}{\underset{x=0}{f'(0) \cdot 4}} = \frac{1}{2}.$$

Therefore by the L'Hospital's Rule $\lim_{x \rightarrow 0} \frac{f(2x+1) - 3}{f(4x)} = \frac{1}{2}$.