QUIZ 4

Question: Find the absolute maximum or minimum values of $f(x) = \sqrt{x}\ln x$ on the interval $(0, e]$ if they exist.

Solution: A function can take its extreme values (if they exist) either at a critical point or a singular point or an end point.

(Note: Please read Theorem 8 (Existence of extreme values on open intervals) Chapter 4, page 238.)

Critical or singular points: Let's consider $f'(x)$ to find the critical or singular points if they exist.

$$f'(x) = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{2+\ln x}{2\sqrt{x}}$$

- $f'(x) = 0 \iff 2 + \ln x = 0 \iff \boxed{x = e^{-2}}$ is the only critical point of f.

- $f'(x)$ is undefined when $x = 0$ but since $0 \notin \text{Domain} = (0, e]$ we DO NOT have any singular point.

x	0		e^{-2}		e
f'		$-$		$+$	
f		↘		↗	
	local max		local min		local max

Since f is decreasing on $(1, e^{-2})$ and is increasing on (e^{-2}, e) and we have only one local min., it is the abs. min. And we have

$$f(e^{-2}) = \sqrt{e^{-2}}\ln(e^{-2}) = \boxed{-\frac{2}{e}}.$$

End points: We are studying on the interval $(0, e]$.

The right end point is included whereas the left end point isn't. That's why we should consider as follows:

- For $x=0$: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \underset{\text{L'H. rule}}{=}$

$$= \lim_{x\to 0^+} -2\sqrt{x} = \boxed{0}$$

- For $x=e$: $f(e) = \sqrt{e}\cdot\ln e = \boxed{\sqrt{e}}$

Conclusion: Comparing the values $0, \sqrt{e}$, and $-\frac{2}{e}$ (among all possible candidates) it is easily seen that f takes its absolute minimum at $x=e^{-2}$ as $f(e^{-2}) = -\frac{2}{e}$ and its absolute maximum at $x=e$ as $f(e)=\sqrt{e}$.

SOME REMARKS FOR GENERAL MISTAKES

- Extreme Value Theorem (EVT) is applicable ONLY for the continuous functions defined on CLOSED and BOUNDED (FINITE) intervals. (Theorem 5, Chp 4, Page 237)

It means that EVT doesn't say anything about the existence of extreme values of a function defined on one of the following intervals:

$$(a,b) \quad \text{or} \quad (-\infty,b) \quad \text{or} \quad (a,+\infty) \quad \text{or} \quad (-\infty,+\infty)$$
$$[a,b) \qquad\qquad (-\infty,b] \qquad\qquad [a,+\infty)$$
$$(a,b]$$

(for above cases, you have to use Thm 8, Chp 4, pg 238.)

- The interval $(0,e]$ is bounded but not closed, that's why EVT cannot be used for our question.

- A point $x_0 \in Dom(f)$ for which $f'(x)$ becomes undefined is called as a singular point of f.

- 0 makes f' undefined but it isn't from $Dom(f)=(0,e]$ so we don't call it a singular point.

- We cannot consider sign table when $x<0$ because $f(x)$ isn't defined for $x<0$. Be careful!

- $f(0)$ is UNDEFINED, so we need to check the limit behaviour of f as x tends to 0^+ to see what happens (f may blow up or approach to some value).