Math 119 2020-1 Recitation week 7

① Find the following - if any - limits.

ⓐ $\lim_{x \to 5} \dfrac{\ln(11-2x)}{x^3-125}$

Solution

$$\lim_{x \to 5} \frac{\ln(11-2x)}{x^3-125} \left[\frac{0}{0}\right] \underset{\text{L'Hopital rule}}{=} \lim_{x \to 5} \frac{\frac{-2}{11-2x}}{3x^2} = \frac{-2}{75}$$

ⓑ $\lim_{x \to 0} \dfrac{x - \sin x}{\ln(1+7x^3)}$

Solution

$$\lim_{x \to 0} \frac{x - \sin x}{\ln(1+7x^3)} \left[\frac{0}{0}\right] \underset{\text{L'Hopital}}{=} \lim_{x \to 0} \frac{1-\cos x}{\frac{21x^2}{1+7x^3}} = \lim_{x \to 0} \frac{(1-\cos x)(1+7x^3)}{21x^2}$$

$$= \lim_{x \to 0} \frac{(1+\cos x)(1-\cos x)(1+7x^3)}{(1+\cos x)\, 21x^2} = \lim_{x \to 0} \frac{\sin^2 x\,(1+7x^3)}{(1+\cos x)\, 21x^2}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1+7x^3}{21(1+\cos x)} \right)$$

$$= \frac{1}{42}$$

ⓒ $\lim_{x \to 0} \dfrac{7^{x^2} - 6^{x^2}}{1-\cos x}$

Solution

$$\lim_{x \to 0} \frac{7^{x^2} - 6^{x^2}}{1-\cos x} \left[\frac{0}{0}\right] \underset{\text{L'Hopital rule}}{=} \lim_{x \to 0} \frac{7^{x^2} \ln 7 . 2x - 6^{x^2} \ln 6 . 2x}{\sin x}$$

$$= \lim_{x \to 0} \frac{2x\,(\ln 7 .\, 7^{x^2} - \ln 6 .\, 6^{x^2})}{\sin x}$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x} \cdot 2(\ln 7 . 7^{x^2} - \ln 6 . 6^{x^2}) \right)$$

$$= (1)\,(2)\,(\ln 7 - \ln 6)$$

$$= 2 \ln\left(\frac{7}{6}\right)$$

(d) $\lim_{x \to 0} (\cos x)^{1/x^2}$

<u>Solution</u>

$$\lim_{x \to 0} (\cos x)^{1/x^2} = \lim_{x \to 0} e^{\ln(\cos x)^{1/x^2}} = \lim_{x \to 0} e^{\frac{\ln(\cos x)}{x^2}}$$

$[\frac{0}{0}]$

$$\underset{\substack{\uparrow \\ e^x \text{ is cont.}}}{=} e^{\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}} \underset{\substack{[\frac{0}{0}] \\ \text{L'Hopital}}}{=} e^{\lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{2x}} = e^{\lim_{x \to 0} \frac{-\sin x}{2x . \cos x}}$$

$$\underset{\substack{[\frac{0}{0}] \\ \text{L'Hopital rule}}}{=} e^{\lim_{x \to 0} \frac{-\cos x}{2\cos x - 2x \sin x}} = e^{-1/2}$$

<u>OR</u>

Let $y = (\cos x)^{1/x^2} \Longrightarrow \ln y = \ln\left((\cos x)^{1/x^2}\right) = \frac{1}{x^2} \ln(\cos x)$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(\cos x)}{x^2} \underset{\text{L'Hopital}}{\overset{[\frac{0}{0}]}{=}} \lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{2x}$$

$$= \lim_{x \to 0} \frac{-\sin x}{2x . \cos x} \underset{\text{L'Hopital}}{\overset{[\frac{0}{0}]}{=}} \lim_{x \to 0} \frac{-\cos x}{2\cos x - 2x \sin x} = -\frac{1}{2}$$

$$-\frac{1}{2} = \lim_{x \to 0} (\ln y) \underset{\substack{\uparrow \\ \ln \text{ is cont.}}}{=} \ln\left(\lim_{x \to 0} y\right) \Rightarrow \lim_{x \to 0} y = e^{-1/2}$$

$$\lim_{x \to 0} y = \lim_{x \to 0} (\cos x)^{1/x^2}$$

② Find all local and absolute extreme points of the following functions:

ⓐ $f(x) = x^4 + 2x^3 - 2x^2 - 6x + 1$

ⓑ $f(x) = \arctan(x - \sqrt{x})$ on $[0,5]$

Solution

ⓐ $f'(x) = 4x^3 + 6x^2 - 4x - 6$

$= 4x(x^2-1) + 6(x^2-1)$

$= (4x+6)(x^2-1) = 0$

$\Rightarrow x = -1,\ x = 1,\ x = -\frac{3}{2}$ are critical points of f

x		$-\frac{3}{2}$		-1		1	
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

At $x = -\frac{3}{2}$ and $x = 1$, f has local minimum

At $x = -1$, f has local max.

So, $\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right)$ and $(1, f(1))$ are local min. points,

and $(-1, f(-1))$ is local max. point of f.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^4\left(1 + \frac{2}{x} - \frac{2}{x^2} - \frac{6}{x^3} + \frac{1}{x^4}\right) = \infty$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x^4\left(1 + \frac{2}{x} - \frac{2}{x^2} - \frac{6}{x^3} + \frac{1}{x^4}\right) = \infty$

So, f has no absolute maximum.

$f(1) = 1 + 2 - 2 - 6 + 1 = -4$

$f\left(\frac{-3}{2}\right) = \frac{81}{16} - \frac{27}{4} - \frac{9}{2} - 9 + 1 \approx 3.81$

$\Rightarrow$ At $x = 1$, f has an absolute minimum value $f(1) = -4$.

So, $(1, f(1)) = (1, -4)$ is the absolute min. point of f.

(b) $f(x) = \arctan(x - \sqrt{x})$ on $[0,5]$

$$f'(x) = \frac{1}{1+(x-\sqrt{x})^2} \cdot (x - \sqrt{x})'$$

$$= \frac{1}{1+(x-\sqrt{x})^2}\left(1 - \frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x}-1}{(1+(x-\sqrt{x})^2)\, 2\sqrt{x}} = 0$$

$$\Rightarrow 2\sqrt{x} - 1 = 0 \quad \text{and} \quad 2\sqrt{x} \neq 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4} \qquad\qquad x \neq 0$$

x	0	$1/4$	5
$f'(x)$	$-$	$+$	
$f(x)$	$\searrow$	$\nearrow$	

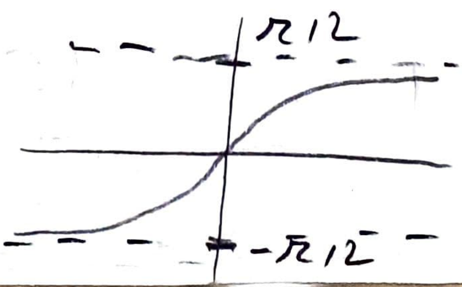
$$f\left(\tfrac{1}{4}\right) = \arctan\left(\tfrac{1}{4} - \tfrac{1}{2}\right) = \arctan\left(-\tfrac{1}{4}\right) = -\arctan\left(\tfrac{1}{4}\right)$$

At $x = \frac{1}{4}$, f has a local minimum, which is $f(\frac{1}{4}) = -\arctan(\frac{1}{4})$.

So, $(\frac{1}{4}, f(\frac{1}{4}))$ is the local minimum point of f.

Since it is the only local min. point of f, and f is continuous on $[0,5]$, then $(\frac{1}{4}, f(\frac{1}{4}))$ is the absolute minimum point of f, $(0, f(0))$ and $(5, f(5))$ are local maximum point of f.

$y = \arctan(x)$



$f(0) = \arctan(0) = 0 < f(5) = \arctan(5 - \sqrt{5})$

So, $(5, f(5))$ is the absolute max. point of f.

3) Find the intervals of concavity of $f(x) = x^2 e^{-x^2}$, and locate any inflection points.

Solution

$$f'(x) = 2xe^{-x^2} + x^2 e^{-x^2}(-2x) = (2x - 2x^3)e^{-x^2}$$

$$f''(x) = (2 - 6x^2)e^{-x^2} + (2x - 2x^3)e^{-x^2}(-2x)$$

$$= (2 - 6x^2 - 4x^2 + 4x^4)e^{-x^2}$$

$$= 2(1 - 5x^2 + 2x^4)e^{-x^2} = 0$$

$$\Rightarrow 2x^4 - 5x^2 + 1 = 0$$

$$\Rightarrow x^2 = \frac{5 \pm \sqrt{25 - 8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

$$\Rightarrow x^2 = \frac{5 - \sqrt{17}}{4}, \quad x^2 = \frac{5 + \sqrt{17}}{4}$$

$$\Rightarrow x_1 = -\frac{\sqrt{5 - \sqrt{17}}}{2}, \quad x_2 = \frac{\sqrt{5 - \sqrt{17}}}{2}, \quad x_3 = -\frac{\sqrt{5 + \sqrt{17}}}{2}, \quad x_4 = \frac{\sqrt{5 + \sqrt{17}}}{2}$$

Note that at an inflection point of f, the concavity changes and f has a tangent line.

x		x_3		x_1		x_2		x_4	
$f''(x)$	$+$		$-$		$+$		$-$		$+$
$f(x)$	$\cup$		$\cap$		$\cup$		$\cap$		$\cup$

$(*)$ Since $f'(x)$ is defined on $\mathbb{R}$, $f'(x_1), f'(x_2), f'(x_3), f'(x_4)$ exist.

On $(-\infty, x_3), (x_1, x_2), (x_4, \infty)$, f is concave up.

On $(x_3, x_1), (x_2, x_4)$, f is concave down.

By $(*)$, f has tangent line at $(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)), (x_4, f(x_4))$. Hence, $(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)), (x_4, f(x_4))$ are inflection points of f.

④ Find all asymptotes of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x+1}$

Solution

Horizontal and Oblique Asymptotes

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x+1} = \lim_{x\to -\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{3x+1}$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{3x+1} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3+\frac{1}{x}\right)} = \frac{\sqrt{2}}{3}$$

$\Rightarrow y = \frac{\sqrt{2}}{3}$ is a horizontal asymptote.

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{3x+1} = \lim_{x\to -\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{3x+1}$$

$$= \lim_{x\to -\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{3x+1} = \lim_{x\to -\infty} \frac{-\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}(3x+1)} = -\frac{\sqrt{2}}{3}$$

$\Rightarrow y = -\frac{\sqrt{2}}{3}$ is a horizontal asymptote.

$\lim_{x\to\pm\infty} f(x)$ are constant $\Rightarrow$ no oblique asymptote

Vertical Asymptotes

$f(x)$ is not defined if $3x+1=0$, i.e. if $x = -\frac{1}{3}$

So, $x = -\frac{1}{3}$ is a candidate for vertical asymptote.

$$\lim_{x\to\left(-\frac{1}{3}\right)^+} f(x) = \lim_{x\to\left(-\frac{1}{3}\right)^+} \frac{\sqrt{2x^2+1}}{3x+1} = +\infty$$

(as $x \to \left(-\frac{1}{3}\right)^+$, $3x+1 \to 0^+$ and $\sqrt{2x^2+1} \to \sqrt{\frac{11}{9}}$)

(OR

$$\lim_{x\to\left(-\frac{1}{3}\right)^-} f(x) = -\infty$$)

So, $x = -\frac{1}{3}$ is a vertical asymptote.

5. Let $f(x)$ be a function which is continuous and differentiable everywhere such that $f(-1)=0$, $f(0)=2$, $f(1)=1$, $f(2)=0$, $f(3)=1$,

and $\lim_{x\to\pm\infty} (f(x)+1-x) = 0$.

Assume that $f'(x)>0$ on $(-\infty,-1)$, $(-1,0)$, $(2,+\infty)$, and that $f'(x)<0$ on $(1,2)$. s.t. $\lim_{x\to -1} f'(x)=\infty$.

Suppose also that $f''(x)>0$ on $(-\infty,-1)$, $(1,3)$, and that $f''(x)<0$ on $(-1,1)$, $(3,+\infty)$.

Sketch the graph of $f(x)$, and identify any critical points.

Solution

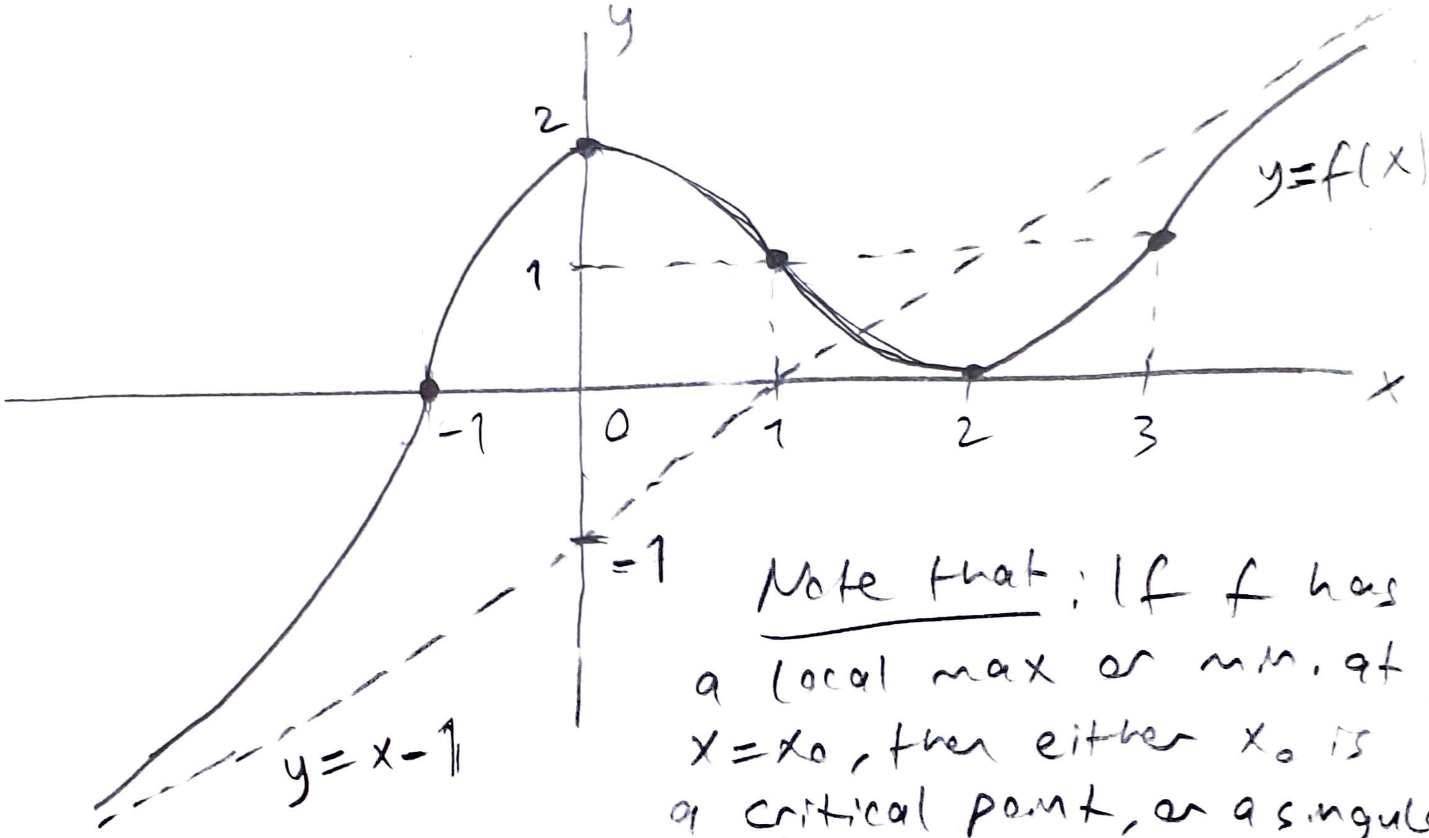
$$0=\lim_{x\to\pm\infty} (f(x)+1-x) = \lim_{x\to\pm\infty} (f(x)-(x-1))$$

So, as $x\to\pm\infty$, $f(x)$ gets closer and closer to $x-1$, i.e. $y=x-1$ is an oblique (inclined) asymptote.

x	-1		0	2	
$f'(x)$	+		+	−	+
$f(x)$	↗		↗	↘	↗

x	-1	1	3	
$f''(x)$	+	−	+	−
$f(x)$	∪	∩	∪	∩

Also, $\lim_{x\to -1} f'(x)=\infty$



<u>Note that</u>: If f has a local max or min. at $x=x_0$, then either x_0 is a critical point, or a singular point or an end point.

f'' exists at $x=0$ and $x=2$. So, f' is continuous at $x=0$ and $x=2$. So, $f'(0)$ and $f'(2)$ exist. Since f has a local max. at $x=0$ and local min. at $x=2$, then $x=0$ and $x=2$ are critical points.