Math 119 2020-1 Recitation Week 6

(1) Evaluate y' for the expression

(a) $y = \tanh(\sin x)$

(b) $y = (\tan x)^{\arctan x}$

(c) $y = 2^{\arcsin(x^3)}$

(d) $\ln(x+y) = \tan^{-1}(xy)$

(e) $y = \arctan(\tanh x)$

(f) $y = \ln(\sinh(2x))$

(g) $y = \dfrac{1+\cosh x}{1-\cosh x}$

Solution

Remember that

- $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- $\dfrac{d}{dx}\cosh x = \sinh x$ and $\dfrac{d}{dx}\sinh x = \cosh x$
- $\tanh x = \dfrac{\sinh x}{\cosh x}$ and $\coth x = \dfrac{\cosh x}{\sinh x}$

(a) $y' = \tanh'(\sin x) \cdot (\sin x)'$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\Rightarrow y' = \operatorname{sech}^2(\sin x) \cdot \cos x$$

(b) $y = (\tan x)^{\arctan x} = e^{\ln(\tan x)^{\arctan x}}$

$$\Rightarrow y = e^{\arctan x \cdot \ln(\tan x)}$$

$$\Rightarrow y' = e^{\arctan x \cdot \ln(\tan x)} \left(\arctan x \cdot \ln(\tan x)\right)'$$

$$= \underbrace{e^{\arctan x \cdot \ln(\tan x)}}_{(\tan x)^{\arctan x}} \left(\frac{1}{1+x^2}\ln(\tan x) + \arctan x \cdot \frac{\sec^2 x}{\tan x}\right)$$

$$\Rightarrow y' = (\tan x)^{\arctan x}\left(\frac{1}{1+x^2}\ln(\tan x) + \arctan x \cdot \frac{1}{\cos x \cdot \sin x}\right)$$

(c) $y = 2^{\arcsin(x^3)} = e^{\ln 2^{\arcsin(x^3)}} = e^{\arcsin(x^3) \cdot \ln 2}$

$$\Rightarrow y' = e^{\arcsin(x^3) \cdot \ln 2} \left(\arcsin(x^3) \cdot \ln 2 \right)'$$

$$= \underbrace{e^{\arcsin(x^3) \cdot \ln 2}}_{\overset{\shortparallel}{2^{\arcsin(x^3)}}} \frac{1}{\sqrt{1 - x^3}} \cdot 3x^2 \cdot \ln 2$$

$$\Rightarrow y' = 2^{\arcsin(x^3)} \ln 2 \, \frac{3x^2}{\sqrt{1 - x^3}}$$

(d) $\ln(x+y) = \tan^{-1}(xy) \; (= \arctan(xy))$

By Implicit Differentiation,

$$\frac{1}{x+y} (x+y)' = \frac{1}{1+x^2y^2} (xy)'$$

$$\Rightarrow \frac{1}{x+y} (1 + y') = \frac{1}{1+x^2y^2} (y + xy')$$

$$\Rightarrow \frac{1}{x+y} + \frac{1}{x+y} y' = \frac{y}{1+x^2y^2} + \frac{x}{1+x^2y^2} \cdot y'$$

$$\Rightarrow y' \cdot \left(\frac{x}{1+x^2y^2} - \frac{1}{x+y} \right) = \frac{1}{x+y} - \frac{y}{1+x^2y^2}$$

$$\Rightarrow y' \cdot \left(\frac{x^2 + xy - 1 - x^2y^2}{\cancel{(1+x^2y^2)(x+y)}} \right) = \frac{1 + x^2y^2 - xy - y^2}{\cancel{(x+y) \cdot (1+x^2y^2)}}$$

$$\Rightarrow y' = \frac{1 + x^2y^2 - xy - y^2}{x^2 + xy - 1 - x^2y^2}$$

(e) $y = \arctan(\tanh x)$

$$\Rightarrow y' = \frac{1}{1+\tanh^2 x} \cdot (\tanh x)' = \frac{1}{1+\tanh^2 x}\left(\frac{\sinh x}{\cosh x}\right)'$$

$$\Rightarrow y' = \frac{1}{1+\tanh^2 x} \cdot \left(\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\right) = \frac{\operatorname{sech}^2 x}{1+\tanh^2 x}$$

(f) $y = \ln(\sinh(2x))$

$$\Rightarrow y' = \frac{1}{\sinh(2x)} \cdot (\sinh(2x))' = \frac{\cosh(2x)}{\sinh(2x)} \cdot 2$$

$$= 2\coth(2x)$$

(g) $y = \dfrac{1+\cosh x}{1-\cosh x}$

$$\Rightarrow y' = \frac{(1+\cosh x)'(1-\cosh x) - (1-\cosh x)'(1+\cosh x)}{(1-\cosh x)^2}$$

$$= \frac{\sinh x(1-\cosh x) - (-\sinh x)(1+\cosh x)}{(1-\cosh x)^2}$$

$$= \frac{\sinh x - \cancel{\sinh x \cosh x} + \sinh x + \cancel{\sinh x \cosh x}}{(1-\cosh x)^2}$$

$$\Rightarrow y' = \frac{2\sinh x}{(1-\cosh x)^2}$$

② Evaluate the following limits without using l'Hôpital Rule.

ⓐ $\lim_{x\to 1} \frac{\ln x}{x-1}$ ⓑ $\lim_{x\to 0} \frac{e^x-1}{x}$

ⓒ $\lim_{x\to -\infty} \arctan\left(\frac{x^2+7x-13}{\sqrt{3}\,x^2+14}\right)$

ⓓ $\lim_{x\to 0} \frac{\cosh x - 1}{x}$ ⓔ $\lim_{x\to\infty} \arctan(x^3-5x+7)$

ⓕ $\lim_{x\to\infty} \frac{2^x+3^x}{3^x+5^x}$

Solution

since $\ln 1 = 0$

ⓐ $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\ln x - \ln 1}{x-1} = \left(\frac{d}{dx}\ln x\right)\Big|_{x=1}$

$= \frac{1}{x}\Big|_{x=1} = 1$

ⓑ $\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x-e^0}{x-0} = \left(\frac{d}{dx}e^x\right)\Big|_{x=0}$

$= e^x\Big|_{x=0} = 1$

ⓒ $\lim_{x\to -\infty} \arctan\left(\frac{x^2+7x-13}{\sqrt{3}x^2+14}\right)$

$= \arctan\left(\lim_{x\to -\infty} \frac{x^2+7x-13}{\sqrt{3}x^2+14}\right)$

$y = \arctan x$ is continuous

$= \arctan\left(\lim_{x\to -\infty} \frac{x^2\left(1+\frac{7}{x}-\frac{13}{x^2}\right)}{x^2\left(\sqrt{3}+\frac{14}{x^2}\right)}\right)$

$= \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(d) $\lim\limits_{x\to 0} \dfrac{\cosh x - 1}{x} = \lim\limits_{x\to 0} \dfrac{\cosh x - \cosh 0}{x - 0}$

$= \dfrac{d}{dx}(\cosh x)\Big|_{x=0} = \sinh 0 = 0$

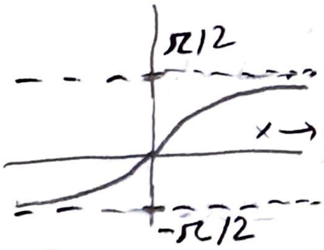
(e) $\lim\limits_{x\to\infty} \arctan(x^3 - 5x + 7)$

$= \arctan\left(\lim\limits_{x\to\infty}(x^3 - 5x + 7)\right)$

y = arctan x is continuous on $\mathbb{R}$

$= \arctan\Big(\lim\limits_{x\to\infty} \underbrace{\underbrace{x^3}_{\infty}\Big(\underbrace{1}_{1} - \underbrace{\cancel{\tfrac{5}{x^2}}}_{0} + \underbrace{\cancel{\tfrac{7}{x^3}}}_{0}\Big)}_{\infty}\Big)$

$= \pi/2$



(DO $\boxed{\text{NOT}}$ WRITE $\arctan(\infty) = \frac{\pi}{2}$)

(f) $\lim\limits_{x\to\infty} \dfrac{2^x + 3^x}{3^x + 5^x} = \lim\limits_{x\to\infty} \dfrac{3^x\left(\left(\frac{2}{3}\right)^x + 1\right)}{5^x\left(\left(\frac{3}{5}\right)^x + 1\right)}$

$= \underbrace{\lim\limits_{x\to\infty} \left(\frac{3}{5}\right)^x}_{= 0} \cdot \underbrace{\lim\limits_{x\to\infty} \dfrac{\left(\frac{2}{3}\right)^x + 1}{\left(\frac{3}{5}\right)^x + 1}}_{= 1} = 0$

since each limit exists

$\Bigg($ $\lim\limits_{x\to\infty} \left(\frac{3}{5}\right)^x = 0$ since $\frac{3}{5} < 1$

$\lim\limits_{x\to\infty} \dfrac{\left(\frac{2}{3}\right)^x + 1}{\left(\frac{3}{5}\right)^x + 1} = \dfrac{\lim\limits_{x\to\infty}\left(\left(\frac{2}{3}\right)^x + 1\right)}{\lim\limits_{x\to\infty}\left(\left(\frac{3}{5}\right)^x + 1\right)} = 1$

since each limit exists $\Bigg)$

(3) A sphere of ice of radius 7 cm is melting at the rate of 12 cm^3 per minute. What is the rate at which its radius is changing when the radius is 5 cm?

Solution

Volume of sphere of ice of radius r is

$V = \frac{4}{3}\pi r^3$, where $V = V(t)$ and $r = r(t)$

$$\frac{dV}{dt} = -12$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$r = 5 \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi . 25} \frac{dV}{dt} = -\frac{12}{100\pi} = -\frac{3}{25\pi}$$

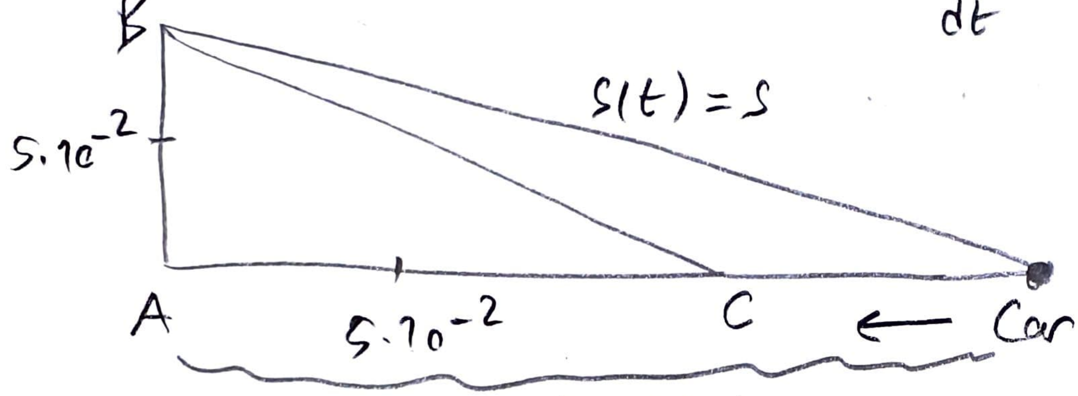
So, the radius is decreasing at the rate of

$\frac{3}{25\pi}$.

(4) ABC is a right isosceles triangle with hypotenuse BC. At the vertex B stands a policeman with a radar gun 50 m away from the highway AC. The policeman points his gun at a car moving through the point C and finds that the distance $|BC|$ is decreasing at the rate of 70 km/h. What is the speed of the car?

Solution

$$\frac{ds}{dt} = -70$$



$x(t) = V \cdot t$

speed of the car

$$s^2 = x^2 + (5 \cdot 10^{-2})^2$$

$$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow s\frac{ds}{dt} = x\frac{dx}{dt}$$

At a time t_0, $x = 5 \cdot 10^{-2}$ and so $s = 5\sqrt{2} \cdot 10^{-2}$

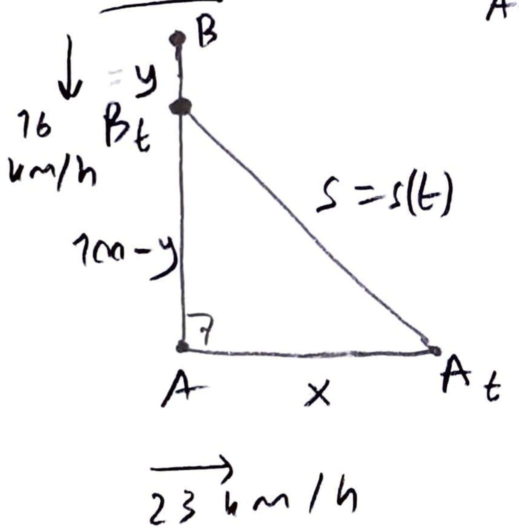
So, $5 \cdot 10^{-2}(-70) = 5 \cdot 10^{-2} \left.\frac{dx}{dt}\right|_{t_0}$

$$\Rightarrow \underbrace{\left.\frac{dx}{dt}\right|_{t_0}}_{\text{velocity at } t_0} = -70\sqrt{2}$$

$$\Rightarrow V = 70\sqrt{2}$$

(5) Captain of the ship A which is moving east at the speed of 23 km/h sees ship B 100 km away in northerly direction moving south at the speed of 16 km/h.
How fast is the distance between the ships will change after half an hour?
How much time elapse until the distance starts to increase?

Soln



At time t, let $s = s(t)$ be the distance between A and B.
let x and y be the distances travelled by A and B, resp.
So, at time t,
$x(t) = 23 \cdot t$ and $y(t) = 16 \cdot t$

$$s^2 = (100 - y)^2 + x^2 = (100 - 16t)^2 + (23 \cdot t)^2$$

$$\Rightarrow 2s\frac{ds}{dt} = 2(100 - 16t)(-16) + 2(23t)\,23$$

$$t = \tfrac{1}{2} \Rightarrow s^2 = \left(100 - 16 \cdot \tfrac{1}{2}\right)^2 + \left(23 \cdot \tfrac{1}{2}\right)^2$$

$$= (92)^2 + \frac{(23)^2}{4} \quad \rightarrow s = \sqrt{(92)^2 + \frac{(23)^2}{4}}$$

and $$2\sqrt{(92)^2 + \frac{(23)^2}{4}}\; \frac{ds}{dt}\Big|_{t=\frac{1}{2}} = -32(100 - 8) + (23)^2$$

$$\Rightarrow \frac{ds}{dt}\Big|_{t=\frac{1}{2}} = \frac{-2415}{2\sqrt{(92)^2 + \frac{(23)^2}{4}}}$$

So, the distance is decreasing at a rate of

$$\frac{2415}{2\sqrt{(92)^2 + \frac{(23)^2}{4}}}$$ after half an hour.

We want $\frac{ds}{dt}$ to be positive so that the distance s between A and B increases:

$$2s\frac{ds}{dt} = 2(100-16t)(-16) + 2(23t)23$$

$$\Rightarrow s\frac{ds}{dt} = (100-16t)(-16) + (23)^2 t$$

$$= \underbrace{-1600 + ((16)^2 + (23)^2)t}_{>0}$$

$$-1600 + ((16)^2 + (23)^2)t > 0 \Rightarrow t > \frac{1600}{(16)^2 + (23)^2}$$

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(6) If a truck factory employs x workers and has daily operating expenses of y dollars, it can produce $P = \frac{x^{0.6}y^{0.4}}{3}$ trucks per year. How fast are the daily expenses decreasing when they are 10,000 dollars and the number of workers is 40, increases at a rate of 1 per day while the production remains constant?

Solution

$$P = \frac{1}{3} x^{0.6} y^{0.4}$$

$$\frac{dP}{dt} = \frac{1}{3}(0.6) x^{-0.4} y^{0.4} \frac{dx}{dt} + \frac{1}{3}(0.4) x^{0.6} y^{-0.6} \frac{dy}{dt}$$

If $\frac{dP}{dt} = 0$, $x = 40$, $y = 10000$, $\frac{dx}{dt} = 1$
production is constant

$$\Rightarrow 0 = (0.2)\, 40^{-0.4} (10^4)^{0.4} \cdot 1 + \frac{(0.4)}{3} 40^{0.6} (10^4)^{-0.6} \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = -375$$

So, daily expenses are decreasing at a rate of 375.