Math119 2020-1 Recitation Problems - Week 5

1. (a) Show that $x^3+x^2+3x+7=0$ has exactly one root.

(b) Let $f(x)=x^4+x^2+x-10$

(i) Show that f has at least 2 roots.

(ii) Show that f does <u>not</u> have 3 roots or more.

<u>Solution</u>

(a) Let $f(x)=x^3+x^2+3x+7$. Then f is cont. on $\mathbb{R}$ since it is a polynomial.

$f(-2)=-8+4-6+7=-3<0$

$f(0)=7>0$

f is cont. on $[-2,0]$. By IVT, $\exists c\in(-2,0)$ s.t. $f(c)=0$. So, f has at least one root.

Suppose f has two roots c_1 and c_2. Suppose, without loss of generality, $c_1<c_2$. Then f is cont. on $[c_1,c_2]$ and diff'ble on (c_1,c_2), since it is a polynomial. By MVT, $\exists m\in(c_1,c_2)$ s.t. $f'(m)=\dfrac{f(c_2)-f(c_1)}{\underbrace{c_2-c_1}_{\neq 0}}=0$.

$f'(x)=3x^2+2x+3 \Rightarrow f'(m)=3m^2+2m+3=0$

But $\Delta=4-4(3)(3)<0 \Rightarrow$ There is <u>no</u> m s.t. $f'(m)=0$, contradiction. So, f has at most one root.

Hence, f has exactly one root.

(b) (i) f is continuous on $\mathbb{R}$ since it is a polynomial.

$f(-2) = 16 + 4 - 2 - 10 = 8 > 0$

$f(-1) = 1 + 1 - 1 - 10 = -9 < 0$

f is cont. on $\mathbb{R} \Rightarrow f$ is cont on $[-2,-1]$.

By IVT, $\exists\, c_1 \in (-2,-1)$ s.t. $f(c_1) = 0$.

$f(1) = 1 + 1 + 1 - 10 = -7 < 0$

$f(2) = 16 + 4 + 2 - 10 = 12 > 0$

f is cont. on $\mathbb{R} \Rightarrow f$ is cont. on $[1,2]$.

By IVT, $\exists\, c_2 \in (1,2)$ s.t. $f(c_2) = 0$.

So, f has at least 2 roots.

(ii) Suppose f has 3 roots, say $c_1 < c_2 < c_3$, without loss of generality. Then,

$f(c_1) = f(c_2) = f(c_3) = 0$.

f is poly. on $\mathbb{R} \Rightarrow f$ is diff'ble on $\mathbb{R}$.

So, f is cont. on $[c_1, c_2]$ and diff'ble on (c_1, c_2),

and f —— " —— $[c_2, c_3]$ —— " —— (c_2, c_3).

By MVT, $\exists\, m_1 \in (c_1, c_2)$ and $\exists\, m_2 \in (c_2, c_3)$ s.t

$f'(m_1) = \dfrac{f(c_2) - f(c_1)}{\underset{\neq 0}{c_2 - c_1}} = 0$ and $f'(m_2) = \dfrac{f(c_3) - f(c_2)}{\underset{\neq 0}{c_3 - c_2}} = 0$

$f'(x) = 4x^3 + 2x + 1 = g(x)$ (call it) $\Rightarrow g'(x) = 12x^2 + 2 > 0\ \forall x$

So, $g(x) = f'(x)$ is strictly increasing. So, $m_1 < m_2 \Rightarrow$

$0 = f'(m_1) < f'(m_2) = 0$, contradiction. So, f does <u>not</u> have 3 or more roots.

② Prove the following inequalities:

ⓐ $e^x \geq x+1 \quad \forall x$

ⓑ $e^x \geq x+1+\frac{x^2}{2} \quad \forall x \geq 0$

Solution

ⓐ If $x=0$, $e^0 = 0+1 \Rightarrow 1=1$

Suppose $x \neq 0$.

Case 1: Suppose $x>0$.

let $f(x) = e^x - x - 1$. $\Rightarrow$ f is cont on $[0,x]$ and diff'ble on $(0,x)$. By MVT, $\exists c \in (0,x)$ s.t. $f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{e^x-x-1}{x}$

$f'(x) = e^x - 1 \Rightarrow f'(c) = e^c - 1 > 0$ ($c>0$)

$\Rightarrow e^x - x - 1 = \underbrace{f'(c)}_{>0}\underbrace{x}_{>0} > 0 \Rightarrow e^x \geq x+1 \quad \forall x>0$

Case 2: Suppose $x<0$.

let $f(x) = e^x - x - 1$. $\Rightarrow$ f is cont. on $[x,0]$ and diff'ble on $(x,0)$. By MVT, $\exists m \in (x,0)$ s.t.

$f'(m) = \frac{f(0)-f(x)}{0-x} = \frac{-e^x+x+1}{-x} = \frac{e^x-x-1}{x}$

$f'(m) = \underbrace{e^m}_{<1} - 1 < 0 \Rightarrow e^x - x - 1 = \underbrace{x}_{<0}\underbrace{f'(m)}_{<0} > 0$

$\Rightarrow e^x \geq x+1 \quad \forall x<0$. Hence, $e^x \geq x+1 \quad \forall x$.

(b) If $x=0$, $e^0 = 0+1+\frac{0^2}{2} \Rightarrow 1=1$

Suppose $x>0$.

Let $f(x) = e^x - x - 1 - \frac{x^2}{2}$. Then f is cont. on $[0,x]$ and diff'ble on $(0,x)$. Then by MVT,

$\exists\, m \in (0,x)$ s.t. $f'(m) = \frac{f(x)-f(0)}{x-0}$.

$$f'(x) = e^x - 1 - x$$

$$\Rightarrow f'(m) = e^m - m - 1 = \frac{e^x - x - 1 - \frac{x^2}{2} - 0}{x-0}$$

$$\Rightarrow e^x - x - 1 - \frac{x^2}{2} = \underbrace{x}_{>0}\,\underbrace{(e^m - m - 1)}_{>0 \text{ by part (a)}} > 0$$

So, $e^x \geq x + 1 + \frac{x^2}{2}$, for $x>0$.

Hence, $e^x \geq x+1+\frac{x^2}{2}$ $\forall x \geq 0$.

(3) Let f be a function defined on $[0,9]$ and f', f'' exist on $(0,9)$.

If $f(1)=-2$, $f(3)=5$, $f(4)=6$, $f(8)=20$, then show that there is a number $c \in (0,9)$ s.t. $f''(c)=0$.

Solution

f' exists on $(0,9) \Rightarrow f$ is continuous on $(0,9)$

By MVT, $\exists\, m_1 \in (1,3)$ s.t. $f'(m_1) = \frac{f(3)-f(1)}{3-1}$

$\Rightarrow f'(m_1) = \frac{5+2}{2} = \frac{7}{2}$

By MVT, $\exists\, m_2 \in (4,8)$ s.t. $f'(m_2) = \frac{f(8)-f(4)}{8-4}$

$\Rightarrow f'(m_2) = \frac{20-6}{8-4} = \frac{7}{2}$

$m_1 \in (1,3)$, $m_2 \in (4,8) \Rightarrow m_1 < m_2$.

f'' exists on $(0,9) \Rightarrow f'$ is cont. on $(0,9)$.

By MVT, $\exists\, c \in (m_1, m_2)$ s.t.

$$f''(c) = \frac{f'(m_2) - f'(m_1)}{m_2 - m_1} = 0.$$

$m_2 - m_1 \neq 0$

④ Given $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$,

ⓐ Show that $f^{-1}(x)$ exists.

ⓑ Find the domain and range of $f^{-1}(x)$.

ⓒ Compute $\frac{df^{-1}}{dx}(6)$.

Solution

ⓐ $f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x)$

$-1 \le \cos(\pi x) \le 1$

$\Rightarrow -2\pi \le -2\pi\cos(\pi x) \le 2\pi$

$\Rightarrow \underbrace{7 - 2\pi}_{0<} \le 7 - 2\pi\cos(\pi x) \le 7 + 2\pi$

$\Rightarrow f'(x) = \underbrace{5x^4}_{\ge 0} + \underbrace{7 - 2\pi\cos(\pi x)}_{>0} > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f$ is increasing on $\mathbb{R}$ $\Rightarrow f$ is 1-1 on $\mathbb{R}$

$\Rightarrow f^{-1}$ exists.

ⓑ $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$ is defined $\forall x \in \mathbb{R}$. So, $\mathrm{Dom}(f) = \mathbb{R}$.

Since the range of f^{-1} is the domain of f, then $\mathrm{Range}(f^{-1}) = \mathbb{R}$. Let's look at $\mathrm{Range}(f)$.

$$\lim_{x \to -\infty} (x^5 + 7x - 2\sin(\pi x) - 2)$$

$$= \lim_{x \to -\infty} x^5\left(1 + \underbrace{\frac{7}{x^4}}_{0} - \underbrace{\frac{2\sin(\pi x)}{x^5}}_{0} - \underbrace{\frac{2}{x^5}}_{0}\right) = -\infty$$

(Show $\displaystyle\lim_{x \to -\infty} \frac{-2\sin(\pi x)}{x^5} = 0$)

$$\lim_{x\to\infty}\left(x^5+7x-2\sin(\pi x)-2\right)$$

$$=\lim_{x\to\infty} x^5\left(1+\underset{\downarrow\, 0}{\frac{7}{x^4}}-\underset{\downarrow\, 0}{\frac{2\sin(\pi x)}{x^5}}-\underset{\downarrow\, 0}{\frac{2}{x^5}}\right)=\infty$$

(show $\lim_{x\to\infty} -\frac{2\sin(\pi x)}{x^5}=0$)

Since f is continuous on $\mathbb{R}$, $\text{Range}(f)=\mathbb{R}$.

So, $\text{Dom}(f^{-1})=\text{Range}(f)=\mathbb{R}$.

(c) Consider $f(f^{-1}(x))=x \quad \forall x\in \text{Range}(f)$

By Implicit Differentiation,

$$f'(f^{-1}(x))\cdot(f^{-1})'(x)=1$$

$$\Rightarrow \frac{df^{-1}}{dx}=(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}$$

$$\Rightarrow \frac{df^{-1}}{dx}(6)=\frac{1}{f'(f^{-1}(6))}$$

$f^{-1}(6)=a \Rightarrow f(a)=6$

$\Rightarrow a^5+7a-2\sin(\pi a)-2=6$

$\Rightarrow a^5+7a-2\sin(\pi a)=8 \Rightarrow a=1$

$\Rightarrow f^{-1}(6)=1$

$$\Rightarrow \frac{df^{-1}}{dx}(6)=\frac{1}{f'(1)}=\frac{1}{12+2\pi}$$

($f'(x)=5x^4+7-2\pi\cos(\pi x)$

$\Rightarrow f'(1)=5+7+2\pi=12+2\pi$)

(5) Find $\frac{dy}{dx}$ at $x=1$ if the differentiable function $y=f(x)$ is defined by
$2xe^y + ye^x = 3e^x$, $f(1)=1$.

By implicit differentiation,

$$2(e^y + xe^y y') + y'e^x + ye^x = 3e^x$$

$x=1, y=1$

$$\Rightarrow 2(e + e y'|_{x=1}) + y'|_{x=1} e + e = 3e$$

$$\Rightarrow \cancel{2e} + 2e y'|_{x=1} + ey'|_{x=1} \cancel{+e} = \cancel{3e}$$

$$\Rightarrow y'|_{x=1} = 0$$

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(6) Consider the curve given by the implicit eqn. $\tan(x+y) = \sin(xy)$. Find the tangent line to this curve at $(\sqrt{\pi}, -\sqrt{\pi})$.

Solution

The tangent line eqn. is
$y-(-\sqrt{\pi}) = m(x-\sqrt{\pi})$ where m is the slope at $(\sqrt{\pi}, -\sqrt{\pi})$.

By implicit differentiation of $\tan(x+y)=\sin(xy)$, we get $\sec^2(x+y)\cdot(x+y)' = \cos(xy)\cdot(xy)'$.

$$\Rightarrow \sec^2(x+y)\cdot(1+y') = \cos(xy)\cdot(y+xy')$$

$x=\sqrt{\pi}$, $y=-\sqrt{\pi}$ $\Rightarrow \underbrace{\sec^2(0)}_{=1}\cdot(1+y'|_{x=\sqrt{\pi}}) = \underbrace{\cos(-\pi)}_{=-1}\cdot(-\sqrt{\pi}+\sqrt{\pi}\, y'|_{x=\sqrt{\pi}})$

$$\Rightarrow 1+y'|_{x=\sqrt{\pi}} = \sqrt{\pi} - \sqrt{\pi}\, y'|_{x=\sqrt{\pi}} \Rightarrow y'|_{x=\sqrt{\pi}} = \frac{\sqrt{\pi}-1}{\sqrt{\pi}+1}$$

$$\Rightarrow y+\sqrt{\pi} = \frac{\sqrt{\pi}-1}{\sqrt{\pi}+1}(x-\sqrt{\pi})$$