Math 119 2020-1 Recitation week 4

1

(a) Consider the curve given by $y = \frac{16}{x} - x^2$;
Find the points where this curve has a horizontal tangent line.

Solution

Let (a,b) be a point on the curve where the curve has a horizontal tangent line. So, the slope of the tangent line to $y = \frac{16}{x} - x^2$ through (a,b) is zero. $\Rightarrow y'|_{x=a} = 0$

$$y' = -\frac{16}{x^2} - 2x \Rightarrow y'|_{x=a} = -\frac{16}{a^2} - 2a = 0$$

$$\Rightarrow \frac{-16-2a^3}{a^2} = 0 \Rightarrow -16-2a^3 = 0 \Rightarrow a = -2$$

$$\Rightarrow b = \frac{16}{a} - a^2 = \frac{16}{-2} - (-2)^2 = -8-4 = -12$$

So, at $(-2,-12)$, the curve has a horizontal tangent line.

(b) For what values of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles?

Solution

$kx^2 = k(x-2)^2 \Rightarrow kx^2 = kx^2 - 4kx + 4k$

$\Rightarrow 4k(-x+1) = 0 \Rightarrow x = 1$

$k \neq 0$ since $y = kx^2$ and $y = k(x-2)^2$ are curves

$x = 1 \Rightarrow y = k(1)^2 = k$

$\Rightarrow$ Intersection points are $(1, k)$.

Slope of the tangent line to $y = kx^2$ at $(1, k)$ is $m_1 = y'\big|_{x=1} = 2kx\big|_{x=1} = 2k$

Slope of the tangent line to $y = k(x-2)^2$ at $(1, k)$ is $m_2 = y'\big|_{x=1} = 2k(x-2)\big|_{x=1}$

$= -2k$

We must have $m_1 \cdot m_2 = -1 \Rightarrow 2k \cdot (-2k) = -1$

$\Rightarrow k^2 = \frac{1}{4} \Rightarrow k = \pm \frac{1}{2}$.

③ Calculate the derivative of the given function using the definition of derivative:

ⓐ $F(x) = \frac{1}{\sqrt{1+x^2}}$ ⓑ $f(x) = x^{1/3}$

Solution

ⓐ $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h}$

if limit exists

$$= \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h\sqrt{1+x^2}\sqrt{1+(x+h)^2}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{1+x^2} - \sqrt{1+(x+h)^2})(\sqrt{1+x^2} + \sqrt{1+(x+h)^2})}{h\sqrt{1+x^2}\sqrt{1+(x+h)^2}(\sqrt{1+x^2} + \sqrt{1+(x+h)^2})}$$

$$= \lim_{h\to 0} \frac{(1+x^2) - (1+(x+h)^2)}{h\sqrt{1+x^2}\sqrt{1+(x+h)^2}(\sqrt{1+x^2} + \sqrt{1+(x+h)^2})}$$

$$= \lim_{h\to 0} \frac{x^2 - x^2 - 2xh - h^2}{h\sqrt{1+x^2}\sqrt{1+(x+h)^2}(\sqrt{1+x^2} + \sqrt{1+(x+h)^2})} = \frac{-x}{(1+x^2)^{3/2}}$$

ⓑ $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h}$

if limit exists

$$= \lim_{h\to 0} \frac{((x+h)^{1/3} - x^{1/3})((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})}{h((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})}$$

$$= \lim_{h\to 0} \frac{(x+h) + \cancel{(x+h)^{2/3}x^{1/3}} + \cancel{(x+h)^{1/3}x^{2/3}} - \cancel{(x+h)^{2/3}x^{1/3}} - \cancel{(x+h)^{1/3}x^{2/3}} - x}{h((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$

(3) How should the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ be defined at $x = 0$ so that it is continuous at $x = 0$? Is it then differentiable there?

Solution

Consider $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

because

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \;\forall x \neq 0 \Rightarrow \underset{\downarrow \, 0}{-x^2} \leq x^2 \sin\left(\frac{1}{x}\right) \leq \underset{\downarrow \, 0}{x^2}$ as $x \to 0$

$\Rightarrow \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$

Squeeze Theorem

Define $F(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\lim_{x \to 0} F(x) = \lim_{x \to 0} f(x) = 0 = F(0)$

So, it is continuous at $x = 0$.

Is it differentiable at 0?

$$\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0}$$

$$= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad \forall x \neq 0$$

If $x > 0$, $\underbrace{-x}_{\downarrow \text{ as } x \to 0^+ \atop 0} \leq x \sin\left(\frac{1}{x}\right) \leq \underbrace{x}_{\downarrow \text{ as } x \to 0^+ \atop 0}$

By squeeze theorem, $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0$.

If $x < 0$, $\underbrace{x}_{\downarrow \text{ as } x \to 0^- \atop 0} \leq x \sin\left(\frac{1}{x}\right) \leq \underbrace{-x}_{\downarrow \text{ as } x \to 0^- \atop 0}$

By squeeze theorem, $\lim_{x \to 0^-} x \sin\left(\frac{1}{x}\right) = 0$.

Hence, $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

So, $\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = 0 = F'(0)$.

Hence, F is differentiable at $x = 0$.

(4) Let $g(x)$ be continuous at $x=a$ and consider the function $f(x)=(x-a)g(x)$. Find $f'(a)$ in terms of g.

Solution

g is given continuous at $x=a$, <u>not</u> differentiable at $x=a$. So, we <u>cannot</u> take derivative of f wrt x directly. Instead, we use the definition of derivative.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\cancel{(x-a)}\,g(x)-0}{\cancel{x-a}}$$

$$=\lim_{x\to a} g(x)=g(a)$$

since g is continuous at $x=a$

---

(5) Given that $f(1)=2$, $f'(1)=1$, $g(1)=3$, $g'(1)=4$, calculate the following:

(a) $\left.\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\right|_{x=1}$ (b) $\left.\frac{d}{dx}\left(x^3 f(x)\right)\right|_{x=1}$

(c) $\left.\frac{d}{dx}\left(f^2(x)g(x)\right)\right|_{x=1}$

Solution

(a) $$\left.\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\right|_{x=1}=\left.\frac{f'(x)(g(x)+x)-(g'(x)+1)f(x)}{(g(x)+x)^2}\right|_{x=1}$$

$$=\frac{f'(1)(g(1)+1)-(g'(1)+1)f(1)}{(g(1)+1)^2}=\frac{1(3+1)-(4+1)2}{(3+1)^2}$$

$$=-6/16=-3/8$$

ⓑ $\frac{d}{dx}\left(x^3 f(x)\right)\Big|_{x=1} = \left(3x^2 f(x) + x^3 f'(x)\right)\Big|_{x=1}$

$= 3f(1) + f'(1) = (3)(2) + 1 = 7$

ⓒ $\frac{d}{dx}\left(f^2(x) g(x)\right)\Big|_{x=1}$

$= \left(\left(\frac{d}{dx}\left(f^2(x)\right)\right) g(x) + f^2(x) \frac{d}{dx} g(x)\right)\Big|_{x=1}$

$= \left(2f(x) f'(x) g(x) + f^2(x) g'(x)\right)\Big|_{x=1}$

$= 2f(1) f'(1) g(1) + f^2(1) g'(1)$

$= (2)(2)(1)(3) + (2)^2(4) = 12 + 16 = 28$

⑥ Find the derivative of the following functions

ⓐ $f(x) = \sqrt{3x + \sqrt{2 + \sqrt{1-x}}}$

ⓑ $g(x) = \left(\frac{1 + \sin(3x)}{3 - 2x}\right)^{-1}$

ⓒ $h(x) = \tan\left(\frac{\pi}{\sqrt{25 - x^2}}\right)$

(Exercise)

Solution

ⓐ Remember that $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$$f'(x) = \frac{1}{2\sqrt{3x + \sqrt{2 + \sqrt{1-x}}}} \cdot \left(3x + \sqrt{2 + \sqrt{1-x}}\right)'$$

chain rule

$$= \frac{1}{2\sqrt{3x + \sqrt{2 + \sqrt{1-x}}}} \cdot \left(3 + \frac{1}{2\sqrt{2 + \sqrt{1-x}}} \cdot \left(2 + \sqrt{1-x}\right)'\right)$$

chain rule

$$\overset{/}{=} \frac{1}{2\sqrt{3x+\sqrt{2+\sqrt{1-x}}}}\cdot\left(3+\frac{1}{2\sqrt{2+\sqrt{1-x}}}\cdot\left(0+\frac{1}{2\sqrt{1-x}}\cdot(1-x)'\right)\right)$$

So, $f'(x) = \frac{1}{2\sqrt{3x+\sqrt{2+\sqrt{1-x}}}}\left(3+\frac{1}{2\sqrt{2+\sqrt{1-x}}}\cdot\frac{-1}{2\sqrt{1-x}}\right)$

(b) $g(x) = \left(\frac{1+\sin(3x)}{3-2x}\right)^{-1}$

$$\Rightarrow g'(x) \underset{/}{=} -\left(\frac{1+\sin(3x)}{3-2x}\right)^{-2}\left(\frac{1+\sin(3x)}{3-2x}\right)'$$

chain rule

$$\underset{/}{=} -\left(\frac{1+\sin(3x)}{3-2x}\right)^{-2}\left(\frac{3\cos(3x)(3-2x)-(-2)(1+\sin(3x))}{(3-2x)^2}\right)$$

quotient rule

(c) $h(x) = \tan\left(\frac{\pi}{\sqrt{25-x^2}}\right)$

$$\Rightarrow h'(x) \underset{/}{=} \sec^2\left(\frac{\pi}{\sqrt{25-x^2}}\right)\cdot\left(\frac{\pi}{\sqrt{25-x^2}}\right)'$$

chain rule

$$= \sec^2\left(\frac{\pi}{\sqrt{25-x^2}}\right)\pi\left((25-x^2)^{-1/2}\right)'$$

$$\underset{/}{=} \pi\sec^2\left(\frac{\pi}{\sqrt{25-x^2}}\right)\cdot\left(-\frac{1}{2}\right)(25-x^2)^{-3/2}\cdot(25-x^2)'$$

chain rule

$$= \pi\sec^2\left(\frac{\pi}{\sqrt{25-x^2}}\right)\left(-\frac{1}{2}\right)(25-x^2)^{-3/2}(-2x)$$

$$= \pi\sec^2\left(\frac{\pi}{\sqrt{25-x^2}}\right)(25-x^2)^{-3/2}\,x$$

(7) (a) Suppose f is a differentiable function and $y = \frac{x}{4} - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at $x = 8$. If $g(x) = (f(x^3))^2$, find an equation for the tangent line to the graph of $y = g(x)$ at $x = 2$.

(b) If $g''(2) = 0$, find $f''(8)$.

Solution

The equation of the tangent line to $y = g(x)$ at $x = 2$ is

$$y - g(2) = \underbrace{g'(2)}_{\text{slope}}(x - 2)$$

$$g(2) = (f(8))^2 = (-1)^2 = 1$$

since $y = \frac{x}{4} - 3$
is tangent to $y = f(x)$
at $x = 8$, $f(8) = \frac{8}{4} - 3 = -1$

$$g'(x) \underset{\text{chain rule}}{=} 2f(x^3).\,(f(x^3))' \underset{\text{chain rule}}{=} 2f(x^3)f'(x^3)\,3x^2$$

So, $g'(2) = 6(2)^2 \underbrace{f(8)}_{= -1} f'(8) = -24f'(8)$

Since $y = \frac{x}{4} - 3$ is tangent to $y = f(x)$ at $x = 8$, $f'(8)$ is the slope of $y = \frac{x}{4} - 3$.

So, $f'(8) = \frac{1}{4}$.

So, $g'(2) = -24 . \frac{1}{4} = -6$

Thus, $y - g(2) = g'(2)(x-2)$

$\Rightarrow y - 1 = -6(x-2) \Rightarrow \boxed{y = -6x + 13}$

(b) By (a), $g'(x) = 6x^2 f(x^3) f'(x^3)$.

$$g''(x) \underset{\text{product rule}}{=} (6x^2)' f(x^3) f'(x^3) + (f(x^3))'\,6x^2 f'(x^3) + (f'(x^3))'\,6x^2 f(x^3)$$

$$= 12x f(x^3) f'(x^3) + f'(x^3).\,3x^2\,6x^2 f'(x^3) + f''(x^3).\,3x^2\,6x^2 f(x^3)$$

$$\Rightarrow g''(2) = 0 = 24f(8)f'(8) + f'(8)(18).(2)^4 f'(8) + f''(8)\,18.(2)^4 f(8)$$

$f(8) = -1$ and $f'(8) = \frac{1}{4} \Rightarrow f''(8) = \frac{1}{24}$