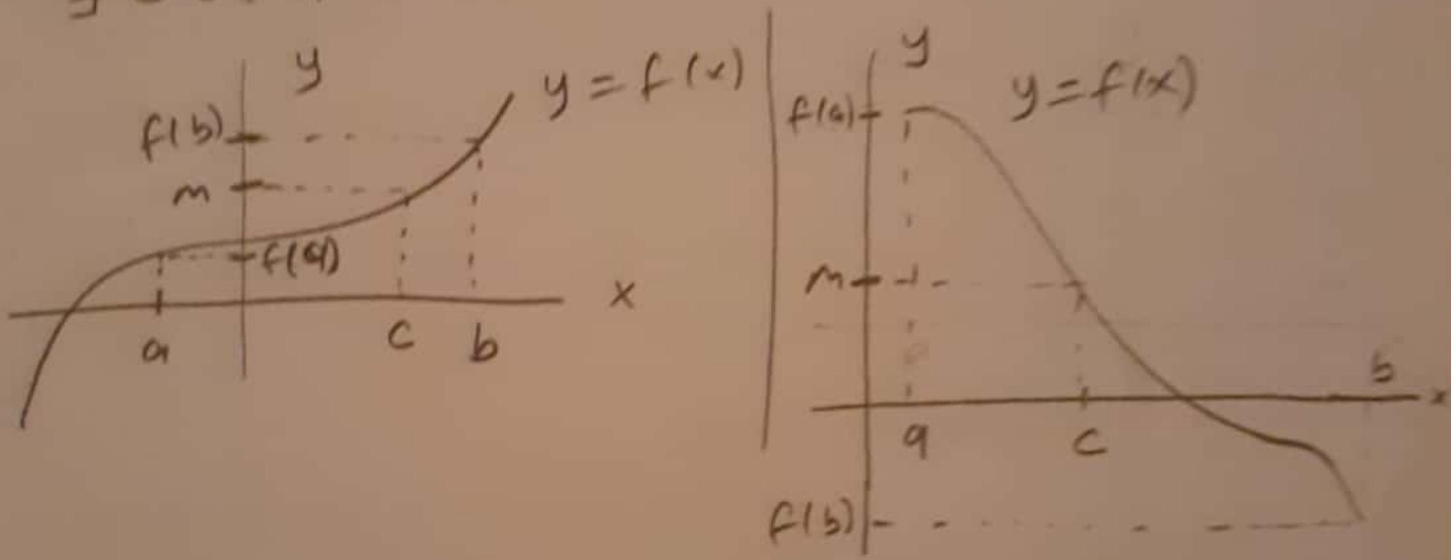
Math 119 2019-1 Recitation week 3

Remember that

① f is continuous at $x=a$ if
$$\lim_{x \to a} f(x) = f(a)$$

② Intermediate Value Theorem (IVT)

If f is continuous on $[a,b]$ and if m is between $f(a)$ and $f(b)$, then $\exists\, c \in (a,b)$ such that $f(c) = m$.



③ "$\lim_{x \to x_0} g(x) = L$ exists" means that

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ s.t. $\forall x$ with
(δ is a function of ε)

$|x - x_0| < \delta$, then $|g(x) - L| < \varepsilon$

④ "$\lim_{x \to x_0} g(x) \neq L$" means that

$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x$ with $|x - x_0| < \delta$,

but $|g(x) - L| \geq \varepsilon$.

Questions

1. Find m so that $g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \geq 3 \end{cases}$

is continuous for all x.

Solution

For $x < 3$, $g(x) = x - m$ is a polynomial, and so, it is continuous.

For $x > 3$, $g(x) = 1 - mx$ is a polynomial, and so, it is continuous.

what about at $x = 3$?

g is continuous at $x = 3$ if $\lim\limits_{x\to 3} g(x) = \underset{\substack{\| \\ 1-3m}}{g(3)}$

$\lim\limits_{x\to 3} g(x)$ exists if $\underset{\substack{\| \\ 3-m}}{\lim\limits_{x\to 3^-} g(x)} = \underset{\substack{\| \\ 1-3m}}{\lim\limits_{x\to 3^+} g(x)}$

$\Rightarrow 1 - 3m = 3 - m \Rightarrow m = -1$

So, for $m = -1$, g is continuous for all x.

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2. Show that there is some a with $0 < a < 2$ s.t. $a^2 + \cos(\pi a) = 4$.

Solution: Let $f(x) = x^2 + \cos(\pi x)$.

$f(0) = 1^2 + \cos(0) = 1$

$f(2) = 2^2 + \cos(2\pi) = 5$

Since f is continuous on $[0,2]$ and since 4 is between $f(0) = 1$ and $f(2) = 5$, by (IVT) Intermediate Value Theorem, $\exists a \in (0,2)$ s.t. $f(a) = a^2 + \cos(\pi a) = 4$.

③ Use the formal definition of the limit to verify the following:

ⓐ $\lim_{x \to c} (ax+b) = ac+b$ for any $a, b, c \in \mathbb{R}$

ⓑ $\lim_{x \to 2} \frac{x-2}{1+x^2} = 0$

ⓒ $\lim_{x \to 3} \sqrt{2x+3} = 3$

<u>Solution</u>

$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ such that $\forall x$ with $|x-x_0|<\delta$ we have $|f(x)-L| < \varepsilon$.

ⓐ <u>If $a=0$</u>, then we need to show that $\lim_{x \to c} b = b$.

Let $\varepsilon > 0$. Then $\forall \delta > 0$ we have that

$|x-c| < \delta \Rightarrow |b-b| = 0 < \varepsilon$.

<u>Suppose $a \neq 0$</u>. Let $\varepsilon > 0$. Find <u>$\delta = \delta(\varepsilon) > 0$</u> such that $|x-c|<\delta \Rightarrow |(ax+b)-(ac+b)| < \varepsilon$.

So, <u>what is δ in terms of ε</u>?

$$|(ax+b)-(ac+b)| = |ax-ac| = |a|\,|x-c| < |a|\,\delta$$

Choose $\delta = \frac{\varepsilon}{|a|} > 0$. Then

$$|x-c|<\delta \Rightarrow |(ax+b)-(ac+b)| = |a|\,|x-c| < |a|\,\delta = |a|\frac{\varepsilon}{|a|} = \varepsilon$$

Hence, $\lim_{x \to c} (ax+b) = ac+b$.

(b) Let $\varepsilon > 0$ be arbitrary.
Find $\delta = \delta(\varepsilon) > 0$ such that

$$|x-2| < \delta \Rightarrow \left| \frac{x-2}{1+x^2} - 0 \right| < \varepsilon.$$

What is δ in terms of ε?

$$\left| \frac{x-2}{1+x^2} - 0 \right| = \frac{|x-2|}{1+x^2} \leq |x-2| < \delta$$

since $1+x^2 \geq 1 \ \forall x \in \mathbb{R}$

Choose $\delta = \varepsilon > 0$. Then

$$|x-2| < \delta \Rightarrow \left| \frac{x-2}{1+x^2} - 0 \right| \leq |x-2| < \delta = \varepsilon.$$

Hence, $\lim_{x \to 2} \frac{x-2}{1+x^2} = 0.$

(c) Let $\varepsilon > 0$ be arbitrary. Find $\delta = \delta(\varepsilon) > 0$
such that $|x-3| < \delta \Rightarrow |\sqrt{2x+3} - 3| < \varepsilon$.

What is δ in terms of ε?

$$|\sqrt{2x+3} - 3| = \left| \frac{(\sqrt{2x+3} - 3)(\sqrt{2x+3} + 3)}{\sqrt{2x+3} + 3} \right|$$

$$= \left| \frac{2x+3-9}{\sqrt{2x+3} + 3} \right| = \frac{2|x-3|}{\sqrt{2x+3} + 3} \leq 2|x-3| < 2\delta$$

$\sqrt{2x+3} + 3 \geq 1$

$\Rightarrow \frac{1}{\sqrt{2x+3} + 3} \leq 1$

Choose $2\delta = \varepsilon$,
i.e. $\delta = \frac{\varepsilon}{2} > 0$.

Then

$|x-3| < \delta = \frac{\varepsilon}{2}$

$$\Rightarrow \left| \frac{(\sqrt{2x+3}-3)(\sqrt{2x+3}+3)}{\sqrt{2x+3}+3} \right| = \frac{2|x-3|}{\sqrt{2x+3}+3}$$

$$\leq 2|x-3| < 2\delta = 2\,\frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim_{x \to 3} \sqrt{2x+3} = 3.$

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(4) Show that the equation $\cos(x) = x^2-1$ has at least two solutions.

Solution

Let $f(x) = \cos(x) - x^2 + 1$. Since $\cos(x)$ and $-x^2+1$ are continuous on $\mathbb{R}$, then $f(x)$ is also continuous on $\mathbb{R}$.

$f(0) = \cos(0) - 0^2 + 1 = 2 > 0$

$f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi^2}{4} + 1 = 1 - \frac{\pi^2}{4} < 0$

$f(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) - \frac{\pi^2}{4} + 1 < 0$

Consider $[-\frac{\pi}{2}, 0]$ and $[0, \frac{\pi}{2}]$.

f is continuous on $[-\frac{\pi}{2}, 0]$, 0 is between $f(-\frac{\pi}{2}) < 0$ and $f(0) > 0$. By Intermediate Value Theorem, there is $c_1 \in (-\frac{\pi}{2}, 0)$ such that $f(c_1) = 0 \Rightarrow \cos(c_1) = c_1^2 - 1$

③

f is continuous on $[0, \frac{\pi}{2}]$ and

0 is between $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$.

By Intermediate Value Theorem, there is

$c_2 \in (0, \frac{\pi}{2})$ such that $f(c_2) = 0$.

$\Rightarrow \cos(c_2) = c_2^2 - 1$.

Since $c_1 \in (-\frac{\pi}{2}, 0)$ and $c_2 \in (0, \frac{\pi}{2})$, $c_1 \neq c_2$

So, $\cos(x) = x^2 - 1$ has at least two solutions

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5 Show that the equation $x^3 - 30x + 2 = 0$ has at least three solutions.

Solution

Let $g(x) = x^3 - 30x + 2$. Then g is continuous on $\mathbb{R}$ since it is a polynomial.

$g(0) = 2 > 0$

$g(1) = 1 - 30 + 2 = -27 < 0$

$$\left.\begin{array}{l} g \text{ is continuous on } [0,1] \\ 0 \text{ is between } g(0) \text{ and } g(1) \end{array}\right\} \xRightarrow{\text{IVT}} \exists c_1 \in (0,1) \text{ such that } g(c_1) = 0.$$

$g(1) = -27 < 0$

$g(6) = 6^3 - 180 + 2 = 38 > 0$

$$\left.\begin{array}{l} g \text{ is cont. on } [1,6] \\ 0 \text{ is between } g(1) \text{ and } g(6) \end{array}\right\} \xrightarrow{\text{IVT}} \exists c_2 \in (1,6) \text{ such that } g(c_2) = 0$$

$g(-6) = -6^3 + 180 + 2 = -34 < 0$

$g(0) = 2 > 0$

$\left.\begin{array}{l} g \text{ is cont. on } [-6,0] \\ 0 \text{ is between } g(-6) \text{ and } g(0) \end{array}\right\} \xRightarrow{IVT} \exists c_3 \in (-6,0)$ such that $g(c_3) = 0$.

$\left.\begin{array}{l} c_1 \in (0,1) \\ c_2 \in (1,6) \\ c_3 \in (-6,0) \end{array}\right\} \Longrightarrow c_1 \neq c_2,\ c_2 \neq c_3,\ c_1 \neq c_3$

So, $g(x) = x^3 - 30x + 2 = 0$ has at least three solutions.

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⑥ Assume that f is a real-valued continuous function such that $\lim\limits_{x \to 0} f(x)\cos^2\left(\frac{\pi}{x}\right) = 0$.

Find $f(0)$.

<u>Solution</u>

f is continuous $\Longrightarrow f(0) = \lim\limits_{x \to 0} f(x)$.

<u>Claim</u>: $f(0) = 0$. Suppose $f(0) = L \neq 0$.

(since f is cont., $\lim\limits_{x \to 0} f(x) = f(0)$ exists)

$$\lim_{x \to 0} \cos^2\left(\frac{\pi}{x}\right) = \lim_{x \to 0} \frac{f(x)\cos^2\left(\frac{\pi}{x}\right)}{f(x)}$$

$$= \frac{\lim\limits_{x \to 0} f(x)\cos^2\left(\frac{\pi}{x}\right)}{\lim\limits_{x \to 0} f(x)} = \frac{0}{L} = 0$$

since each limit exists and $f(0) \neq 0$

contradiction since $\lim\limits_{x \to 0} \cos^2\left(\frac{\pi}{x}\right)$ does <u>not</u> ex

$\boxed{\text{Hence, } f(0) = 0.}$

⑦ Suppose $f(x)$ is continuous on $\mathbb{R}$. ⑤

Let $f(0)=1$, $f(1)=-1$, $f(-1)=0$.

Show that $\exists c \in \mathbb{R}$ such that $f(f(c))=c$.

<u>Solution</u>

Define $g(x)=f(f(x))-x$.

Since f is continuous on $\mathbb{R}$, then g is also continuous on $\mathbb{R}$.

$g(-1)=f(f(-1))-(-1)=f(0)+1=2>0$

$g(1)=f(f(1))-1=f(-1)-1=0-1=-1<0$

Since 0 is between $g(-1)$ and $g(1)$,

by IVT, $\exists c \in (-1,1)$ such that $g(c)=0$.

So, $f(f(c))=c$. $\qquad g(c) = f(f(c))-c$

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⑧ Show that the equation $1-\frac{x^2}{4}=\cos x$ has at least

ⓐ one real solution

ⓑ two real solutions

ⓒ three real solutions

<u>Solution</u>

Define $h(x)=\underbrace{1-\frac{x^2}{4}}_{\text{cont. on } \mathbb{R}}-\underbrace{\cos x}_{\text{cont. on } \mathbb{R}} \Rightarrow h$ is cont on $\mathbb{R}$

ⓐ $h(0)=1-0-1=0$. So, $x=0$ is a real solution of $1-\frac{x^2}{4}=\cos x$.

(b) $h\left(\frac{\pi}{2}\right) = 1 - \underbrace{\frac{\pi^2}{16}}_{<1} - \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} > 0$

$h\left(\frac{3\pi}{2}\right) = 1 - \underbrace{\frac{9\pi^2}{16}}_{>1} - \underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0} < 0$

Since h is continuous on $\mathbb{R}$, then it is continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Since 0 is between $h\left(\frac{\pi}{2}\right)$ and $h\left(\frac{3\pi}{2}\right)$, then by IVT, $\exists\, c \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such that

$h(c) = 1 - \frac{c^2}{4} - \cos(c) = 0.$

Since $0 \notin \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $1 - \frac{x^2}{4} = \cos(x)$ has at least two real solutions, which are 0 (by part (a)) and c.

(c) $h(-x) = 1 - \frac{(-x)^2}{4} - \cos(-x) = 1 - \frac{x^2}{4} - \cos(x)$
$= h(x)$

So, h is even, i.e. the graph of h is symmetric with respect to y-axis.
Thus, $1 - \frac{x^2}{4} = \cos(x)$ has at least three real solutions, which are

$0,\ c \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),\ -c \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right).$