Math 119 2020-1 Recitation Week 2

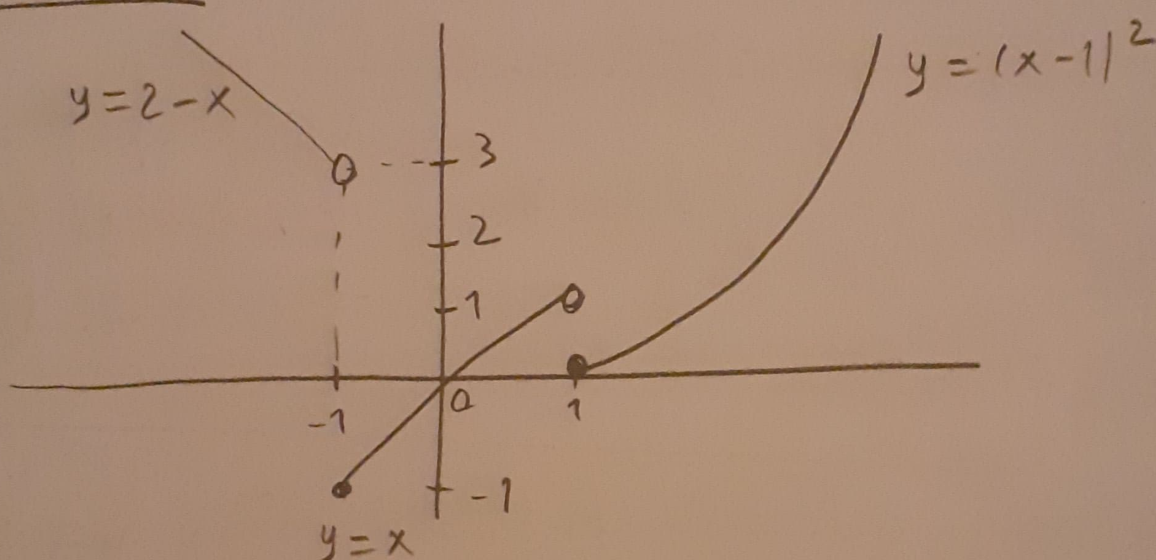
① Sketch the graph of

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ (x-1)^2 & \text{if } x \ge 1 \end{cases}$$

and find the limits

ⓐ $\lim_{x \to -1} f(x)$ ⓑ $\lim_{x \to 0} f(x)$ ⓒ $\lim_{x \to 1} f(x)$

Solution



ⓐ $\lim_{x \to -1^-} f(x) = 3 \neq \lim_{x \to -1^+} f(x) = -1 \Rightarrow \lim_{x \to -1} f(x)$ does <u>not</u> exist

ⓑ $\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^-} f(x) \Rightarrow \lim_{x \to 0} f(x) = 0$

ⓒ $\lim_{x \to 1^-} f(x) = 1 \neq \lim_{x \to 1^+} f(x) = 0 \Rightarrow \lim_{x \to 1} f(x)$ does <u>not</u> exist

② Let $f(x) = \frac{|x-2|}{x-2}$. Find ⓐ $\lim_{x \to 0} f(x)$

ⓑ $\lim_{x \to 2} f(x)$

Solution

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -x+2 & \text{if } x < 2 \end{cases}$$

ⓐ Near 0, $|x-2| = -x+2$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-x+2}{x-2} = -1$$

ⓑ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{-x+2}{x-2} = -1$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$

not equal

So, $\lim_{x \to 2} f(x)$ does not exist.

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③ Evaluate the limits, if they exist.

ⓐ $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$ ⓑ $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$

ⓒ $\lim_{t \to 0} \frac{(3+t)^{-1} - 3^{-1}}{t}$ ⓓ $\lim_{x \to 1} \frac{\sqrt{x} - x^2}{1 - \sqrt{x}}$

Solution

ⓐ $$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \lim_{x \to 1} \frac{x^2+x+1}{x+1} = 3/2$$

ⓑ $$\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \to 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)} = \lim_{h \to 0} \frac{(\sqrt{1+h})^2 - 1^2}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)} = \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2}$$

ⓒ $\lim_{t\to 0} \dfrac{(3+t)^{-1}-3^{-1}}{t} = \lim_{t\to 0} \dfrac{\frac{1}{3+t}-\frac{1}{3}}{t}$

$= \lim_{t\to 0} \dfrac{-t}{3t(3+t)} = \lim_{t\to 0} \dfrac{-1}{3(3+t)} = -\dfrac{1}{9}$

ⓓ $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}-(\sqrt{x})^4}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}\,(1-(\sqrt{x})^3)}{1-\sqrt{x}}$

$= \lim_{x\to 1} \dfrac{\sqrt{x}\,\cancel{(1-\sqrt{x})}\,(1+\sqrt{x}+(\sqrt{x})^2)}{\cancel{1-\sqrt{x}}} = 1.(1+1+1)$

$= 3$

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④ Let $f(x) = x - \lfloor x \rfloor$.

ⓐ Sketch the graph of f.

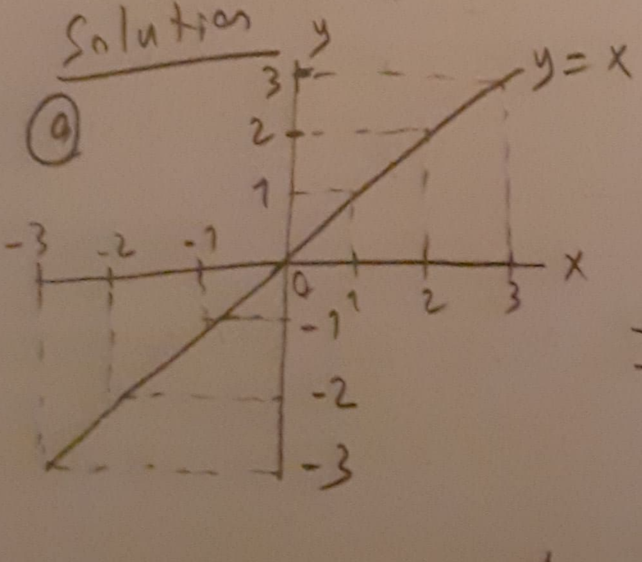
ⓑ If n is an integer, find (i) $\lim_{x\to n^-} f(x)$

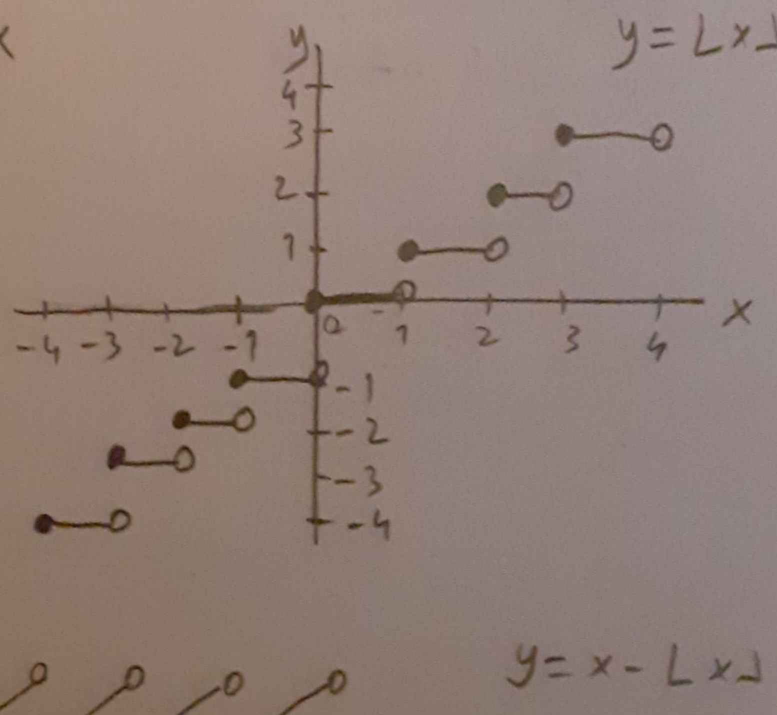
(ii) $\lim_{x\to n^+} f(x)$

ⓒ For what values of a, does $\lim_{x\to a} f(x)$ exist?

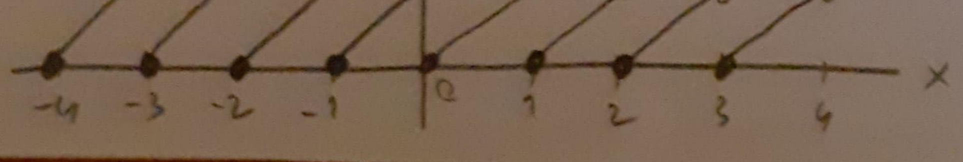
Solution

ⓐ





$\Rightarrow$



(b) Suppose n is an integer. Then

(i) $\lim_{x \to n^-} f(x) = \lim_{x \to n^-} (x - \lfloor x \rfloor) = n - (n-1) = 1$

at the LHS of n, $\lfloor x \rfloor = n - 1$

(ii) $\lim_{x \to n^+} f(x) = \lim_{x \to n^+} (x - \lfloor x \rfloor) = n - n = 0$

at the RHS of n, $\lfloor x \rfloor = n$

(c) By part (b), if a is an integer, then

$\lim_{x \to a^-} f(x) = 1 \neq \lim_{x \to a^+} f(x) = 0.$

So, if a is an integer, $\lim_{x \to a} f(x)$ does <u>not</u> exist.

Suppose a is <u>not</u> an integer.
then by the graph of $f(x) = x - \lfloor x \rfloor$,
$f(x)$ has the same value at the RHS and LHS of a.

So, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Thus, $\lim_{x \to a} f(x)$ exists if a is <u>not</u> an integer.

(5) If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

Solution

$f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = 0$

since 2 and -2 are integers

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \lfloor x \rfloor + \lim_{x\to 2^-} \lfloor -x \rfloor = 1 + \lim_{x\to 2^-} \lfloor -x \rfloor$

since each limit exists

$= 1 + \lim_{t\to -2^+} \lfloor t \rfloor = 1 + (-2) = -1$

(Let $t = -x$
$x \to 2^- \Rightarrow t \to -2^+$)

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \lfloor x \rfloor + \lim_{x\to 2^+} \lfloor -x \rfloor = 2 + \lim_{x\to 2^+} \lfloor -x \rfloor$

since each limit exists

$= 2 + \lim_{t\to -2^-} \lfloor t \rfloor = 2 + (-3) = -1$

(Let $t = -x$
$x \to 2^+ \Rightarrow t \to -2^-$)

So, $\lim_{x\to 2} f(x) = -1 \neq f(2) = 0$.

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(6) Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$

Solution: $\lim_{x\to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} = \lim_{x\to 2} \frac{(\sqrt{6-x} - 2)(\sqrt{3-x} + 1)}{(\sqrt{3-x} - 1)(\sqrt{3-x} + 1)}$

$= \lim_{x\to 2} \frac{(\sqrt{6-x} - 2)(\sqrt{3-x} + 1)}{2-x} = \lim_{x\to 2} \frac{(\sqrt{3-x} + 1)(\sqrt{6-x} - 2)(\sqrt{6-x} + 2)}{(2-x)(\sqrt{6-x} + 2)}$

$= \lim_{x\to 2} \frac{(\sqrt{3-x} + 1)\cancel{(2-x)}}{(\sqrt{6-x} + 2)\cancel{(2-x)}} = \frac{1}{2}$

⑦ Prove that

ⓐ $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$ ⓑ $\lim_{h \to 0^+} \sqrt{h}\, e^{\sin\left(\frac{\pi}{h}\right)} = 0$

Solution

ⓐ $-1 \le \cos\left(\frac{2}{x}\right) \le 1 \quad \forall x \ne 0$

$\Rightarrow -x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4 \quad \forall x > 0$

$-x^4 \to 0$ as $x \to 0^+$, $x^4 \to 0$ as $x \to 0^+$

By squeeze Theorem, $\lim_{x \to 0^+} x^4 \cos\left(\frac{2}{x}\right) = 0$

Also, $-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4 \quad \forall x < 0$

$-x^4 \to 0$ as $x \to 0^-$, $x^4 \to 0$ as $x \to 0^-$

By squeeze Theorem, $\lim_{x \to 0^-} x^4 \cos\left(\frac{2}{x}\right) = 0$

Hence, $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$.

ⓑ $-1 \le \sin\left(\frac{\pi}{h}\right) \le 1 \quad \forall h \ne 0$

e^x is increasing $\Rightarrow e^{-1} \le e^{\sin\left(\frac{\pi}{h}\right)} \le e \quad \forall h \ne 0$

$\Rightarrow \sqrt{h}\, e^{-1} \le \sqrt{h}\, e^{\sin\left(\frac{\pi}{h}\right)} \le \sqrt{h}\, e \quad \forall h > 0$

$\sqrt{h}\, e^{-1} \to 0$ as $h \to 0^+$, $\sqrt{h}\, e \to 0$ as $h \to 0^+$

By squeeze Theorem, $\lim_{h \to 0^+} \sqrt{h}\, e^{\sin\left(\frac{\pi}{h}\right)} = 0$

⑧ Evaluate $\lim\limits_{x \to \pm\infty} \dfrac{1}{\sqrt{x^2-2x}-2}$.

Solution

$$\lim_{x \to \pm\infty} \frac{1}{\sqrt{x^2-2x}-2} = \lim_{x \to \pm\infty} \frac{1}{|x|\sqrt{1-\frac{2}{x}}-2}$$

$$= \lim_{x \to \pm\infty} \frac{1}{|x|\left(\sqrt{1-\frac{2}{x}}-\frac{2}{|x|}\right)}$$

$$= \underbrace{\lim_{x \to \pm\infty} \frac{1}{|x|}}_{=0} \cdot \underbrace{\lim_{x \to \pm\infty} \frac{1}{\sqrt{1-\frac{2}{x}}-\frac{2}{|x|}}}_{=1} = 0$$

since each limit exists

⑨ Evaluate $\lim\limits_{x \to \infty} \left(\dfrac{x^2}{x+1} - \dfrac{x^2}{x-1}\right)$.

Solution

$$\lim_{x \to \infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right) = \lim_{x \to \infty} \frac{x^2(x-1) - x^2(x+1)}{(x+1)(x-1)}$$

$$= \lim_{x \to \infty} \frac{x^3 - x^2 - x^3 - x^2}{x^2-1} = \lim_{x \to \infty} \frac{-2x^2}{x^2-1}$$

$$= \lim_{x \to \infty} \frac{-2\cancel{x^2}}{\cancel{x^2}\left(1-\frac{1}{x^2}\right)} = \lim_{x \to \infty} \frac{-2}{1-\frac{1}{x^2}} = -2$$