Math119 2020-1 Recitation

① Solve the following inequalities:

ⓐ $\dfrac{1}{2x+1} \geq 1-x$, ⓑ $|x+3|-2>3x$ ⓒ $\dfrac{x^3-x^2+4}{x+3} \leq 1$

Solution

ⓐ $\dfrac{1}{2x+1} \geq 1-x \Rightarrow \dfrac{1-(1-x)(2x+1)}{2x+1} \geq 0$

$\Rightarrow \dfrac{1-(2x+1-2x^2-x)}{2x+1} \geq 0 \Rightarrow \dfrac{2x^2-x}{2x+1} \geq 0 \Rightarrow \dfrac{x(2x-1)}{2x+1} \geq 0$

$x=0,\ x=\frac{1}{2},\ x \neq -\frac{1}{2}$

x		$-\frac{1}{2}$		0		$\frac{1}{2}$	
x	$-$		$-$	0	$+$		$+$
$2x-1$	$-$		$-$		$-$	0	$+$
$2x+1$	$-$	0	$+$		$+$		$+$
$\frac{x(2x-1)}{2x+1}$	$-$		$+$		$-$		$+$

$\Rightarrow$ The solution set is $\left(-\frac{1}{2}, 0\right] \cup \left[\frac{1}{2}, \infty\right)$

ⓑ $|x+3| = \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$

If $x \geq -3$, $|x+3|-2 > 3x \Rightarrow x+3-2 > 3x$

$\Rightarrow 1 > 2x \Rightarrow x < \frac{1}{2} \Rightarrow \left[-3, \frac{1}{2}\right)$

If $x < -3$, $|x+3|-2 > 3x \Rightarrow -x-3-2 > 3x$

$\Rightarrow -5 > 4x \Rightarrow x < -\frac{5}{4} \Rightarrow (-\infty, -3)$

$\Rightarrow$ The solution set is $(-\infty, -3) \cup \left[-3, \frac{1}{2}\right)$

$= \left(-\infty, \frac{1}{2}\right)$

ⓒ $\frac{x^3-x^2+4}{x+3} \le 1 \Rightarrow \frac{x^3-x^2+4-x-3}{x+3} \le 0$

$\Rightarrow \frac{x^3-x^2-x+1}{x+3} \le 0$

$x^3-x^2-x+1=0$

$x=1 \Rightarrow 1-1-1+1=0 \Rightarrow x^3-x^2-x+1=(x-1)\ldots?$

$$\begin{array}{r|l} x^3-x^2-x+1 & x-1 \\ \cline{2-2} -x^3-x^2 & x^2-1 \\ \hline -x+1 & \\ --x+1 & \\ \hline 0 & \end{array}$$

$\Rightarrow x^3-x^2-x+1=(x-1)^2(x+1)$
$=0$

$\Rightarrow x=-1,\ x=1,\ x=1$

Also, $x \neq -3$

x		-3		-1		1	
$x+1$	$-$		$-$	0	$+$		$+$
$(x-1)^2$	$+$		$+$		$+$	0	$+$
$x+3$	$-$	0	$+$		$+$		$+$
$\frac{x^3-x^2-x+1}{x+3}$	$+$		$-$		$+$		$+$

$\Rightarrow$ The solution set is $(-3,-1] \cup \{1\}$.

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② Write an equation for the line through the points $(-1,5)$ and $(0,3)$.

Solution: let $(x_0,y_0)=(-1,5)$, $(x_1,y_1)=(0,3)$.

The eqn. of line is $y-y_0=m(x-x_0)$

or $y-y_1=m(x-x_1)$.

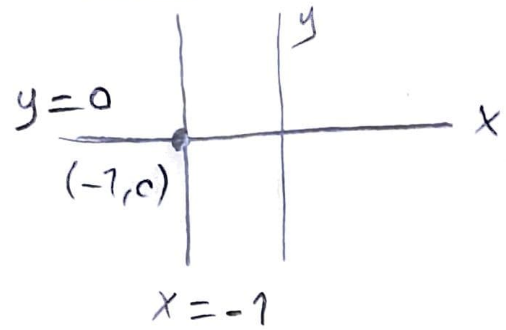
slope $= m = \frac{y_0-y_1}{x_0-x_1} = \frac{5-3}{-1-0} = -2$

$\Rightarrow$ the eqn. of the line is $y-3=-2(x-0)$

$\Rightarrow y=-2x+3.$

3. Find the eqn. for the
(a) vertical line (b) horizontal line
through the point $(-1,0)$.

Solution



(a) Vertical line through $(-1,0)$ is
$x=-1$

(b) Horizontal line through $(-1,0)$ is
$y=0$

4. Find the eqn. for the line through $P(-1,3)$ that is perpendicular to the line $y+x+2=0$.
Find x and y intercepts of this line.

Solution

$l_1: y+x+2=0 \Rightarrow y=-x-2 \Rightarrow m_{l_1}=-1$

$l_2: y-3=m_{l_2}(x-(-1))$

Since $l_1 \perp l_2$, $\underbrace{m_{l_1}}_{=-1} \cdot m_{l_2}=-1 \Rightarrow m_{l_2}=1$

$\Rightarrow l_2: y-3=1(x+1) \Rightarrow y=x+4$

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5. Describe and sketch the regions defined by the followings

(a) $x^2+y^2-2x-4y \le 4$

(b) $x^2+y^2 \le 4$, $x^2+y^2 > 2y$

(c) $x^2+y^2 > 2y$, $y > 1+x$

(d) $y > (x-1)^2+2$, $y < 2x$

(e) $4x^2+(y-2)^2 \le 4$

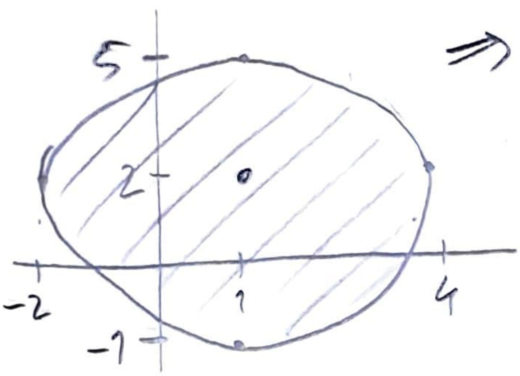
exercise

Solution

ⓐ $x^2+y^2-2x-4y \le 4$

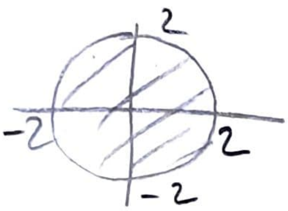
$\Rightarrow (x^2-2x+\underline{\underline{1}})+(y^2-4y+\underline{\underline{4}}) \le 4+5=9$

$\Rightarrow (x-1)^2+(y-2)^2 \le 9 = 3^2$



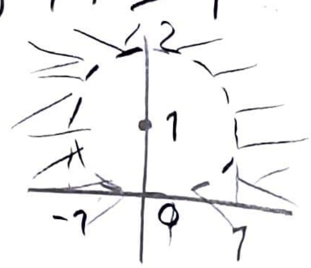
$\Rightarrow$ Inside and boundary of the circle with center $(1,2)$ and radius 3.

ⓑ $x^2+y^2 \le 4 \Longrightarrow$

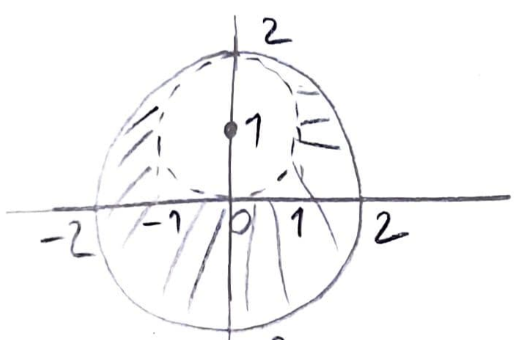


$x^2+y^2>2y \Rightarrow x^2+y^2-2y+1>1$

$\Rightarrow x^2+(y-1)^2>1 \Longrightarrow$



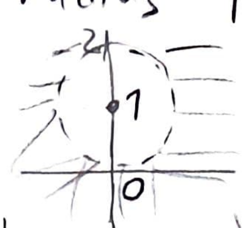
$\Rightarrow$



$\Rightarrow$ Inside and boundary of the circle with center $(0,0)$ and radius 2 and outside of circle with center $(0,1)$ and radius 1

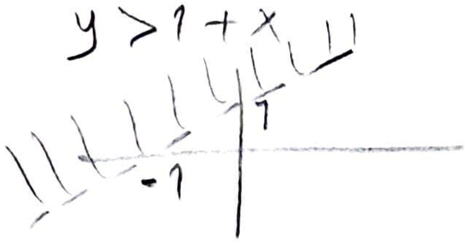
ⓒ $x^2+y^2>2y$

$\Rightarrow x^2+(y-1)^2>1 \Longrightarrow$



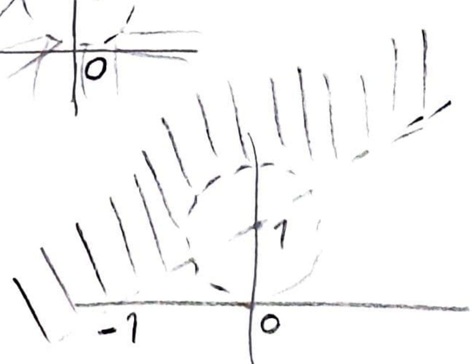
ⓑ

$y>1+x$



$\Longrightarrow$

$\Downarrow$



(6) Find the points of intersection of the pairs of curves

(a) $y=x^2+3$, $y=3x+1$, (b) $2x^2+2y^2=5$, $xy=1$

(Exercise)

<u>Solution</u>

(b) Let (a,b) be a point of intersection of $2x^2+2y^2=5$ and $xy=1$. $\Rightarrow 2a^2+2b^2=5$ and $ab=1$.

$ab=1 \Rightarrow b=\frac{1}{a}$)

$2a^2+2b^2=5 \Longrightarrow 2a^2+\frac{2}{a^2}=5 \Rightarrow 2a^4-5a^2+2=0$

$\Rightarrow (2a^2-1)(a^2-2)=0 \Rightarrow a^2=\frac{1}{2}, a^2=2$

$\Longrightarrow a=\pm\frac{1}{\sqrt{2}}$, $a=\pm\sqrt{2}$

Use $\underline{\underline{ab=1}}$ or $2a^2+2b^2=5$:

$a=-\frac{1}{\sqrt{2}} \Rightarrow b=-\sqrt{2}$, $a=\frac{1}{\sqrt{2}} \Rightarrow b=\sqrt{2}$

$a=-\sqrt{2} \Rightarrow b=-\frac{1}{\sqrt{2}}$, $a=\sqrt{2} \Rightarrow b=\frac{1}{\sqrt{2}}$

$\Rightarrow \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right), \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right), \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

are the points of intersections of $2x^2+2y^2=5$ and $xy=1$.

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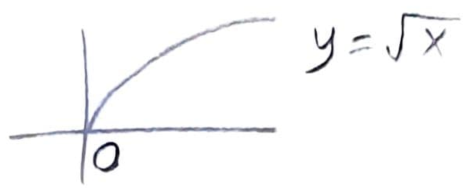
(7) Write an eqn. of the graph obtained by shifting the graph of $y=\sqrt{x}$.

(a) down 1, right 1 (b) down 2, left 4

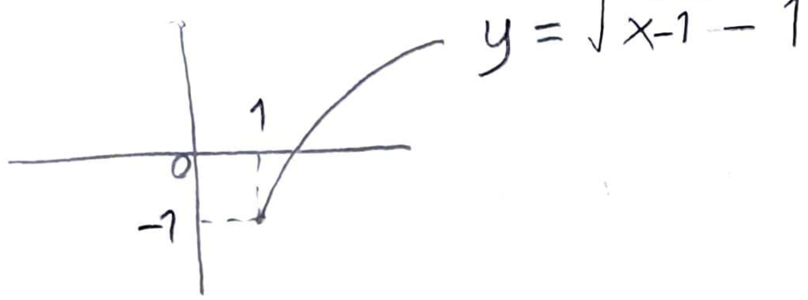
(c) up 2, left 1 (d) up 1, right 1

b, d exercise

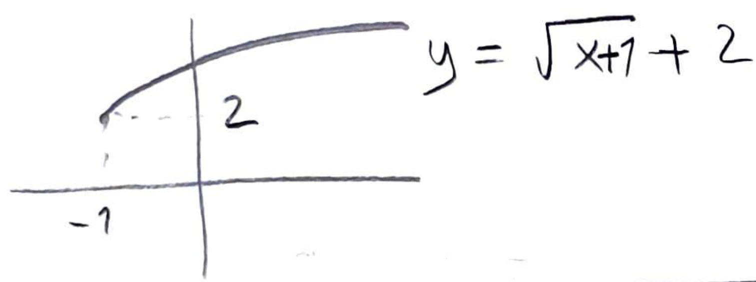
Solution :



ⓐ We will shift the graph of $y=\sqrt{x}$ down 1 and right 1 unit.



ⓒ We will shift the graph of $y=\sqrt{x}$ up 2 and left 1 unit.



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② Find the domain and range of each function and sketch their graphs:

ⓐ $f(x)=\sqrt{x^2-1}$, ⓑ $f(x)=\dfrac{1}{|2-x|}$, ⓒ $y=1+\sin\left(x+\frac{\pi}{4}\right)$
EX

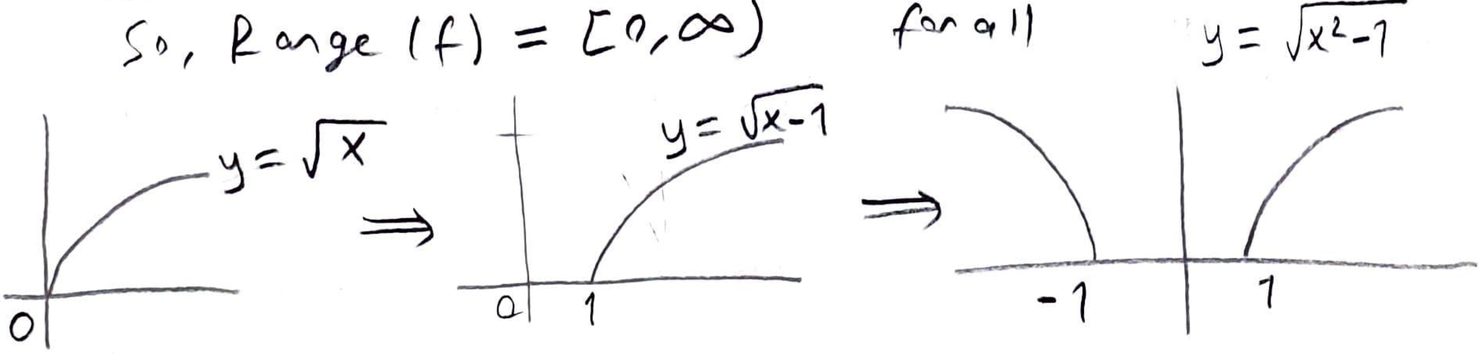
Solution

ⓐ Domain : $x^2-1\geq 0 \Rightarrow x^2\geq 1 \Rightarrow x\geq 1$ or $x\leq -1$

$Dom(f)=(-\infty,-1]\cup[1,\infty)$

Range : $f(x)=\sqrt{x^2-1}\geq 0 \ \ \forall x\in Dom(f)$
for all

So, $Range(f)=[0,\infty)$



(b) $f(x) = \dfrac{1}{|2-x|} = \dfrac{1}{|x-2|}$

<u>Domain</u>: $|2-x| \neq 0 \Rightarrow 2-x \neq 0 \Rightarrow x \neq 2$

$\mathrm{Dom}(f) = \mathbb{R} - \{2\}$

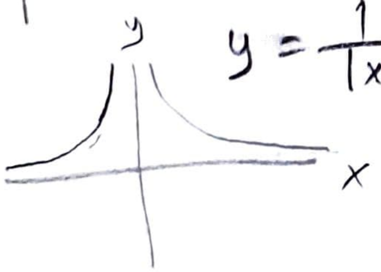
<u>Range</u>: $|\ldots| \geq 0 \Rightarrow \dfrac{1}{|\ldots|} > 0$

$\Rightarrow f(x) = \dfrac{1}{|2-x|} > 0 \quad \forall x \in \mathrm{Dom}(f)$

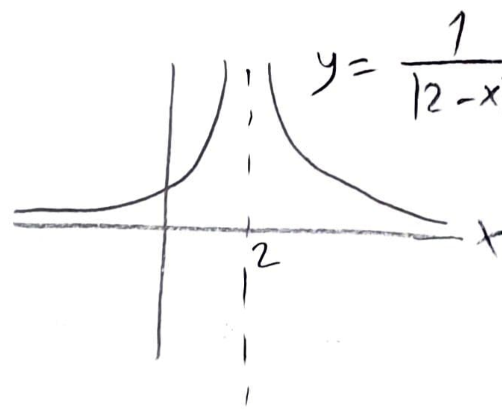
$\Rightarrow \mathrm{Range}(f) = (0, \infty)$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\Rightarrow$



$\Rightarrow$



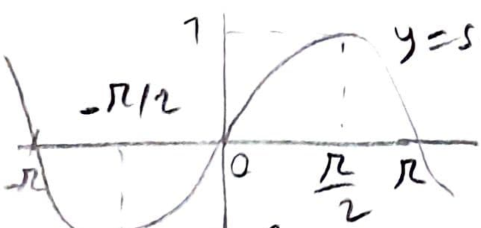
(c) $y = f(x) = 1 + \sin\left(x + \frac{\pi}{4}\right)$

<u>Domain</u>: $\sin(x)$ is defined $\forall x \in \mathbb{R}$

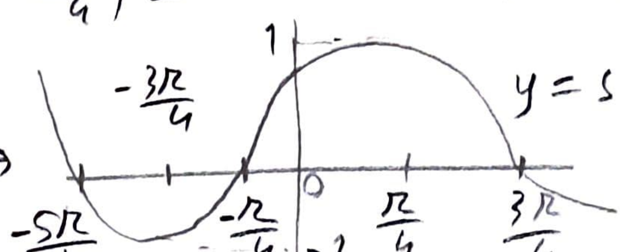
$\Rightarrow \sin\left(x + \frac{\pi}{4}\right)$ — " — $\Rightarrow \mathrm{Dom}(f) = \mathbb{R}$

<u>Range</u>: $-1 \leq \sin\left(x + \frac{\pi}{4}\right) \leq 1 \quad \forall x \in \mathbb{R}$

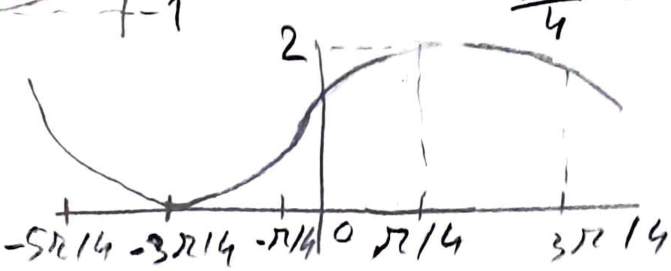
$\Rightarrow 0 \leq 1 + \sin\left(x + \frac{\pi}{4}\right) \leq 2 \quad \forall x \in \mathbb{R} \Rightarrow \mathrm{Range}(f) = [0, 2]$



$\Rightarrow$



$\Rightarrow$



$y = 1 + \sin\left(x + \frac{\pi}{4}\right)$

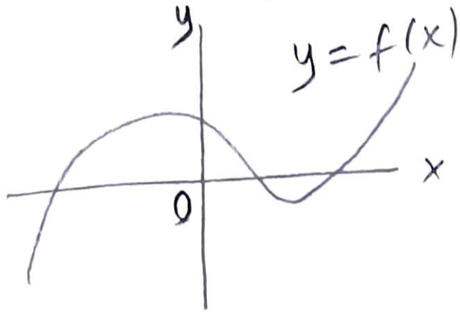
9. (a) How is the graph of $y = f(|x|)$ related to the graph of f?

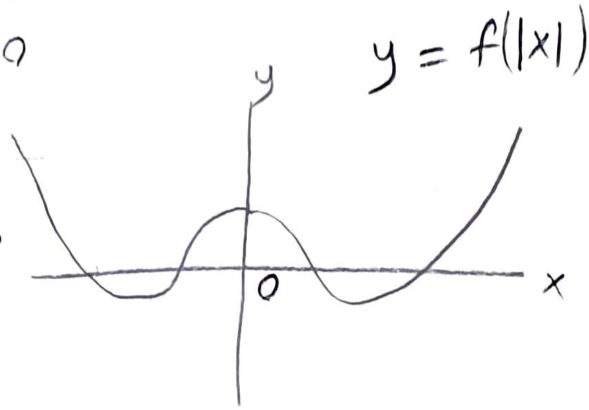
(b) Sketch the graph of $y = \sin|x|$.

(c) Sketch the graph of $y = \sqrt{|x|}$.

Solution

(a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



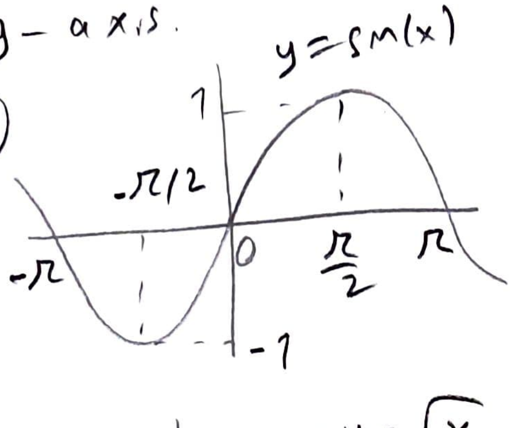


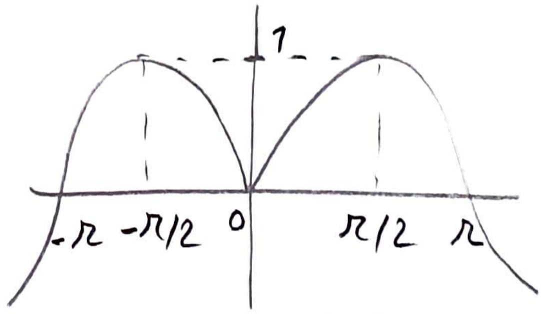
After and at 0, $|x| = x$. So, after 0, $f(|x|) = f(x)$.

Before 0, $|x| = -x$. So, $f(|x|) = f(-x)$

So, we just take the symmetry with respect to y-axis.

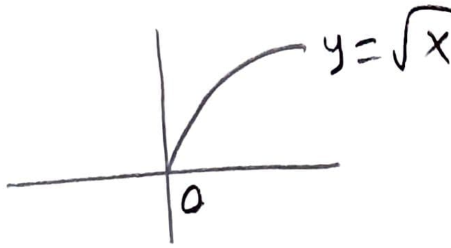
(b)

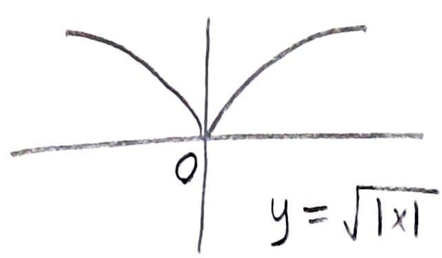




$y = \sin|x|$

(c)





(10) ⓐ Find $f \circ g$ and its domain where

$f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x-1}{x+3}$.

ⓑ Given $F(x) = \sin^2(x-5)$, find functions f, g and h s.t. $F = f \circ g \circ h$.

<u>Solution</u>

ⓐ $(f \circ g)(x) = f(g(x)) = g(x) + \frac{1}{g(x)} = \frac{x-1}{x+3} + \frac{1}{\frac{x-1}{x+3}}$

$\Rightarrow (f \circ g)(x) = \frac{x-1}{x+3} + \frac{x+3}{x-1}$

$Dom(f) = \mathbb{R} - \{-3, 1\}$

ⓑ Take $f(x) = x^2$, $g(x) = \sin(x)$, $h(x) = x - 5$.

$\Rightarrow F(x) = (f \circ g \circ h)(x) = f(g(h(x)))$

where $g(h(x)) = \sin(x-5)$

$\Rightarrow F(x) = (\sin(x-5))^2 = \sin^2(x-5)$.