Math 501 Homework 2

Due Date November 29, 2021 Monday

November 22, 2021

Chapter 4

Problem 1

Construct a set $B \subset [0, 1]$ with Lebesgue measure $1/3$ which is not Borel measurable.

Chapter 5

Recall that we say a function $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is measurable if $\forall B \in \mathcal{B}$ we have $f^{-1}(B) \in \mathcal{A}$.

Problem 1

Give an example of a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ which is not Borel measurable.

Problem 2

Let $f : X \to (Y, \mathcal{B})$ be a function where $(Y, \mathcal{B})$ is a measurable space. Let $\mathcal{A} = \{A \subset X | A = f^{-1}(B) \text{ for some } B \in \mathcal{B}\}$.

(a) Show that $\mathcal{A}$ is a σ-algebra.

(b) Show that $\mathcal{A}$ is the smallest σ-algebra on X making f measurable.

Problem 3

Let $f : (X, \mathcal{A}) \to Y$ be a function where $(X, \mathcal{A})$ is a measurable space. Let $\mathcal{B} = \{B \subset Y | f^{-1}(B) \in \mathcal{A}\}$.

(a) Show that $\mathcal{B}$ is a σ-algebra.

(b) Show that $\mathcal{B}$ is the largest σ-algebra on Y making f measurable.

Problem 4

Let $(X, \mathcal{A})$ be a measurable space and let $f_n : (X, \mathcal{A}) \to \mathbb{R}$ be measurable function for each $n = 1, 2, 3, \cdots$. Suppose we can write X as a disjoint union $X = \cup_{n=1}^{\infty} A_n$ where each $A_n \in \mathcal{A}$. Now let $f : X \to \mathbb{R}$ be defined so that $f|_{A_n} = f_n|_{A_n}$ for all $n = 1, 2, 3, \cdots$. Show that f is measurable.

Problem 5

Given $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ define $d(x, A) = \inf\{|x - y| \ : y \in A\}$ "minimum distance of A from x". Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} d(x, \mathbb{P}) & x \in \mathbb{Q} \\ d(x, \mathbb{Z}) & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{P}$ is the set of prime numbers. Show that f is measurable.