Math 501 Homework 1

Due Date November 12, 2021 Friday

November 5, 2021

Chapter 2

Problem 1

Let $\mathcal{A}$ be a $\sigma - algebra$. Given $A_1, A_2, A_3, \cdots \in \mathcal{A}$ define

$$B = \{x \in X \mid x \text{ belongs to exactly 3 of } A_i\text{'s}\}$$

Show that $B \in \mathcal{A}$.

Problem 2

Let $X = \mathbb{N} = \{1, 2, 3, \cdots\}$ and let $k \cdot \mathbb{N} = \{k, 2k, 3k, \cdots\}$.

(a) What is the $\sigma - algebra$ generated by the collection $\{k \cdot \mathbb{N} \mid k \in \mathbb{N}\}$?

(b) What is the $\sigma - algebra$ generated by the collection $\{p \cdot \mathbb{N} \mid p \text{ is a prime number}\}$?

Chapter 4

In problems 1 and 2, m is the *Lebesgue measure*. In problem 3, m^* is a general outer measure. In problem 4, m^* is the outer measure associated to the α function given.

Problem 1

Show that $\forall r \in \mathbb{R}$ the singleton $\{r\}$ is *Lebesgue measurable*. What is $m(\{r\})$?

Problem 2

Show that the set $J = [0,1] - \mathbb{Q}$ is *Lebesgue measurable*. What is $m(J)$?

Problem 3

Let m^* be an outer measure on X. Given $A \subset B \subset C \subset X$ where A and C are $m^* - measurable$ sets with $m^*(A) = m^*(C)$. Show that B is also $m^* - measurable$ with $m^*(B) = m^*(A) = m^*(C)$.

Problem 4

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be defined as

$$\alpha(x) = \lfloor x \rfloor \text{ (greatest integer less than or equal to } x)$$

(a) Show that α is increasing and right-continuous.

(b) Give a formula for $m^*(A)$ for any $A \subset \mathbb{R}$.

(c) What is the $\sigma - algebra$ of $m^* - measurable$ subsets?