Math 120 (Spring 2021)

Sections 21-22

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Recitation week 3

Def:

Absolute convergence

The series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm: If a series converges absolutely, then it converges.

Def:

Conditional convergence

If $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say that it is **conditionally convergent** or that it **converges conditionally**.

Thm:

The alternating series test

Suppose $\{a_n\}$ is a sequence whose terms satisfy, for some positive integer N,

(i) $a_n a_{n+1} < 0$ for $n \geq N$,

(ii) $|a_{n+1}| \leq |a_n|$ for $n \geq N$, and $\Rightarrow |a_{n+1}| - |a_n| \leq 0$ $\hookrightarrow \frac{|a_{n+1}|}{|a_n|} \leq 1$

(iii) $\lim_{n\to\infty} a_n = 0$,

that is, the terms are ultimately alternating in sign and decreasing in size, and the sequence has limit zero. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

Thm:

Error estimate for alternating series

If the sequence $\{a_n\}$ satisfies the conditions of the alternating series test (Theorem 14), so that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum s, then the error in the approximation $s \approx s_n$ (where $n \geq N$) has the same sign as the first omitted term $a_{n+1} = s_{n+1} - s_n$, and its size is no greater than the size of that term:

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|.$$

$\Rightarrow |\text{error}| \leq |a_{n+1}|$

1. Determine whether the given series is absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\sqrt[3]{5n^7+n-1}}$

Firstly, we look at the series

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n n}{\sqrt[3]{5n^7+n-1}} \right| = \sum_{n=2}^{\infty} \frac{n}{\sqrt[3]{5n^7+n-1}}$$

$$\frac{n}{\sqrt[3]{5n^7+\underbrace{n-1}_{\geq 0}}} \leq \frac{n}{\sqrt[3]{5n^7}} = \frac{n}{\sqrt[3]{5}\, n^{7/3}} = \frac{1}{\sqrt[3]{5}\, n^{4/3}}$$ and

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{5}\, n^{4/3}} = \frac{1}{\sqrt[3]{5}} \sum_{n=2}^{\infty} \frac{1}{n^{4/3}}$$ is convergent by p-test $\left(p = \frac{4}{3} > 1\right)$

So $\sum_{n=2}^{\infty} \frac{n}{\sqrt[3]{5n^7+n-1}}$ is also convergent by comparison

Then $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\sqrt[3]{5n^7+n-1}}$ is absolutely convergent

(b) $\displaystyle\sum_{n=18}^{\infty} \frac{(-1)^n n}{4n^2 - 1000}$

a_n

If we look at this series

$$\sum_{n=18}^{\infty} |a_n| = \sum_{n=18}^{\infty} \frac{n}{4n^2-1000}$$

↳ it's positive when $n \geq 18$

By the LCT with $\displaystyle\sum_{n=18}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n}{4n^2-1000}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{4n^2-1000} \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1}{4-\frac{1000}{n^2}}$$

$$= \frac{1}{4} > 0$$

and $\displaystyle\sum_{n=18}^{\infty} \frac{1}{n}$ is divergent, so $\displaystyle\sum_{n=18}^{\infty} \frac{n}{4n^2-1000}$ is divergent

$\to \displaystyle\sum_{n=18}^{\infty} a_n$

By the Alternating Series Test (A.S.T)

i) $$a_n \cdot a_{n+1} = \frac{(-1)^n n}{4n^2-1000} \cdot \frac{(-1)^{n+1}(n+1)}{4(n+1)^2-1000}$$

$$= \frac{\overset{-}{(-1)^{2n+1}} \, \overset{+}{n} \, \overset{+}{(n+1)}}{\underset{+}{(4n^2-1000)} \, \underset{+}{(4(n+1)^2-1000)}} < 0$$

for all $n \geq 18$

ii) $|a_{n+1}| - |a_n| = \dfrac{n+1}{4(n+1)^2-1000} - \dfrac{n}{4n^2-1000}$

$4n^2+8n-996$

$$= \frac{\cancel{4n^3} + 4n^2 - 1000n - 1000 - \cancel{4n^3} - 8n^2 + 996n}{(4(n+1)^2-1000)(4n^2-1000)}$$

$$= \frac{-4n^2 - 4n - 1000}{(4(n+1)^2-1000)(4n^2-1000)}$$

$\Delta = 16 - 4.4.1000 < 0$

$-4 < 0$

$-$

$+ \qquad +$

$< 0 \Rightarrow |a_{n+1}| - |a_n| < 0$

$\Rightarrow |a_{n+1}| < |a_n|$

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{(-1)^n n}{4n^2-1000} \cdot \dfrac{1/n^2}{1/n^2} = \lim_{n\to\infty} \dfrac{\cancel{\dfrac{(-1)^n}{n}}}{4 - \cancel{\dfrac{1000}{n^2}}}$

$= 0$ ✓

Therefore, from (i), (ii) and (iii),

$\sum_{n=18}^{\infty} \dfrac{(-1)^n n}{4n^2-1000}$ is convergent

$\sum_{n=18}^{\infty} a_n$ converges but $\sum_{n=18}^{\infty} |a_n|$ diverges $\Rightarrow \sum a_n$ converges conditionally

$$(c) \sum_{n=3}^{\infty} \underbrace{\frac{(-n)^n n}{(2n)!}}_{a_n}$$

If we look at the series

$$\sum_{n=3}^{\infty} |a_n| = \sum_{n=3}^{\infty} \underbrace{\frac{n^{n+1}}{(2n)!}}_{b_n}$$

By the ratio test

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+2}}{(2n+2)!}}{\frac{n^{n+1}}{(2n)!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)}(n+1)^{n+1}}{\cancel{(2n)!}(2n+1)\underbrace{\cancel{(2n+2)}}_{2}} \cdot \frac{\cancel{(2n)!}}{n^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{4n+2} \cdot \left(\frac{n+1}{n}\right)^{n+1} = \lim_{n\to\infty} \underbrace{\frac{1}{4n+2}}_{0} \cdot \underbrace{\left(1+\frac{1}{n}\right)^n}_{e} \cdot \underbrace{\left(1+\frac{1}{n}\right)}_{1}$$

$$= 0 = p \Rightarrow 0 \leq p < 1$$

$$\Rightarrow \sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} b_n \text{ is convergent}$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n \text{ is abs convergent}$$

Exercise

$$(d)\ \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n e^{1/n}}{n^3}}_{a_n}$$

Look at the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n e^{1/n}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$$

$\frac{e^{1/n}}{n^3} < \frac{e}{n^3}$ and $\sum_{n=1}^{\infty} \frac{e}{n^3}$ is convergent by p-test $(p = 3 > 1)$

so $\sum |a_n| = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ is convergent by comparison

$\Rightarrow \sum a_n$ is abs convergent

$$(e)\ \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{(\arctan n)^n}}_{a_n}$$

Look at the series

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{(\arctan n)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\arctan(n))^n}} = \lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

and $0 < \frac{2}{\pi} < 1$

then $\sum |a_n|$ is convergent by the root test

So $\sum a_n$ is abs convergent

2. How many terms of the given series do we need to add in order to find the sum to the indicated accuracy?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{\underbrace{n!}_{b_n}}$ with $|Error| \leq 10^{-3}$

i) it's alternating series

ii) $\frac{|b_{n+1}|}{|b_n|} = \frac{\frac{\cancel{2^{n+1}}\,2}{\cancel{(n+1)!}\,n+1}}{\frac{\cancel{2^n}}{\cancel{n!}}} = \frac{2}{n+1} \leq 1$ for all $n \geq 1$

$\frac{|b_{n+1}|}{|b_n|} \leq 1 \Rightarrow |b_{n+1}| \leq |b_n|$ ($|b_n|$ is decreasing)

iii) $\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \left| \frac{(-1)^n 2^n}{n!} \right| = \lim_{n\to\infty} \frac{2^n}{n!} = 0$

$$0 < \frac{2^n}{n!} = \underbrace{\frac{2}{n}}_{} \cdot \underbrace{\underbrace{\frac{2}{n-1}}_{\leq 1} \cdots \underbrace{\frac{2}{2}}_{\leq 1} \cdot \underbrace{\frac{2}{1}}_{\leq 1}}_{\leq 1} \leq \underbrace{\frac{2}{n} \cdot 2}_{}$$

$0 \to 0$, $\frac{2}{n}\cdot 2 \to 0$

then, by the squeeze thm $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$\Rightarrow \sum_{n=1}^{\infty} b_n$ satisfies the conditions of AST $\Rightarrow \sum_{n=1}^{\infty} b_n$ converges

Let's find N s.t $|error| \leq |b_{N+1}| \leq 10^{-3}$

$|b_7| = \frac{2^7}{7!} = \frac{8}{315} \nleq 10^{-3}$

$|b_8| = \frac{2^8}{8!} = \frac{2}{315} \nleq 10^{-3}$

$|b_9| = \frac{2^9}{9!} = \frac{4}{2835} \nleq 10^{-4}$

$|b_{10}| = \frac{2^{10}}{10!} = \frac{4}{14175} \leq 10^{-3}$

so $N+1 \geq 10 \Rightarrow N \geq 9$

at least first 9 terms

$|error| \leq 10^{-3}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with $|Error| \leq 0,002$

(Lecture 8 Mehmetcik Pamuk at 42:30)

Def:

Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is called a **power series in powers of $x - c$** or a **power series about c**. The constants $a_0, a_1, a_2, \ldots$ are called the **coefficients** of the power series.

Thm:

For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ one of the following alternatives must hold:

(i) the series may converge only at $x = c$,

(ii) the series may converge at every real number x, or

(iii) there may exist a positive real number R such that the series converges at every x satisfying $|x - c| < R$ and diverges at every x satisfying $|x - c| > R$. In this case the series may or may not converge at either of the two *endpoints* $x = c - R$ and $x = c + R$.

In each of these cases the convergence is absolute except possibly at the endpoints $x = c - R$ and $x = c + R$ in case (iii).

3. Determine the center, radius, and the interval of convergence of the given power series

(a) $\sum_{n=0}^{\infty} \underbrace{n!(x+2)^n}_{a_n}$ the center is $x=-2$

By the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x+2)^{n+1}}{n!\,(x+2)^n}\right| = \lim_{n\to\infty}(n+1)\,|x+2| < 1$$

this limit equal to ∞ except at $x=-2$

$\Rightarrow$ converges if $x=-2$

diverges if $x\neq-2$

$\Rightarrow$ the radius of convergence is $R=0$

" interval " " is $\{-2\}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ the center is $x=2$

$\to a_n$

By the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{3^{n+1}}}{\frac{(x-2)^n}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{x-2}{3}\right| = \left|\frac{x-2}{3}\right| < 1$$

$\Rightarrow$ divergent if $\left|\frac{x-2}{3}\right| > 1$

convergent if $\left|\frac{x-2}{3}\right| < 1 \Rightarrow |x-2| < 3$

$\Rightarrow -3 < x-2 < 3$

$\Rightarrow -1 < x < 5$

$\Rightarrow (-1, 5)$

$\Rightarrow R = \frac{6}{2} = 3$

For the end points,

i) at $x=-1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges

ii) at $x=5$, $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1$ diverges

So the interval of convergence is $(-1, 5)$

(c) $\sum_{n=0}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$ $\rightarrow a_n$ the center is $x = \frac{-3}{2}$

By the ratio test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(2x+3)^{n+1}}{\sqrt{n+2}}}{\frac{(2x+3)^n}{\sqrt{n+1}}} \right|$$

$$= \lim_{n\to\infty} |2x+3| \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} = |2x+3|$$

$\Rightarrow$ divergent if $|2x+3| > 1$

convergent if $|2x+3| < 1$

$\Rightarrow -1 < 2x+3 < 1$

$\Rightarrow -4 < 2x < -2$

$\Rightarrow -2 < x < -1$

$\Rightarrow (-2, -1)$

$\Rightarrow R = \frac{1}{2}$ //

For the end points

i) at $x = -2$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \to a_n$

$\to$ it's alternating series

$\to$ $|a_{n+1}| = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = |a_n|$ ($|a_n|$ is decreasing)

$\to$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0$

$\Rightarrow$ it's convergent by A.S.T.

ii) at $x = -1$, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

by the LCT with $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges

$\Rightarrow$ the interval of convergence is $[-2, -1)$

Exercise

(d) $\sum_{n=3}^{\infty} \underbrace{\frac{(-1)^n x^{3n}}{\ln n}}_{a_n}$ The center is $x = 0$

By the ratio test,

$= \ln(n) + \ln(1+\frac{1}{n})$

$= \ln(n(1+\frac{1}{n}))$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1} x^{3n+3}}{\ln(n+1)}}{\frac{(-1)^n x^{3n}}{\ln n}}\right| = \lim_{n\to\infty} |x^3| \cdot \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} |x^3| \frac{\ln n}{\ln n + \ln(1+\frac{1}{n})}$$

$= |x^3|$

$\Rightarrow$ divergent if $|x^3| > 1 \Rightarrow |x| > 1$

$\Rightarrow$ convergent if $|x^3| < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow (-1, 1) \Rightarrow R = \frac{2}{2} = 1$

For the end points,

i) at $x = -1$, $\sum_{n=3}^{\infty} \frac{1}{\ln(n)}$ diverges (From the question 5-b of week 2)

ii) at $x = 1$, $\sum_{n=3}^{\infty} \underbrace{\frac{(-1)^n}{\ln(n)}}_{b_n}$ $\rightarrow$ $\ln(n) > 0$ for $n \geq 3$

$\rightarrow$ it's alternating

$\rightarrow |b_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = |b_n|$ since $\ln$ is increasing function

$\rightarrow \lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\Rightarrow$ this series is convergent by the AST

$\Rightarrow$ the interval of convergence is $(-1, 1]$