Math 120 (Spring 2021)

Sections 71 - 72

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Recitation week 2

1. Find the sum of the series $\sum_{k=1}^{\infty} \ln\left(\frac{\arctan(k+1)}{\arctan k}\right)$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{\arctan(k+1)}{\arctan k}\right)$$

$$= \lim_{n\to\infty} \left[\ln\left(\frac{\arctan(2)}{\arctan 1}\right) + \ln\left(\frac{\arctan 3}{\arctan 2}\right) + \dots + \ln\left(\frac{\arctan(n+1)}{\arctan(n)}\right)\right]$$

$$= \lim_{n\to\infty} \ln\left[\frac{\cancel{\arctan 2}}{\arctan 1} \cdot \frac{\cancel{\arctan 3}}{\cancel{\arctan 2}} \cdots \frac{\arctan(n+1)}{\cancel{\arctan n}}\right]$$

$$= \lim_{n\to\infty} \ln\left[\frac{\arctan(n+1)}{\arctan 1}\right] = \ln\left[\frac{\overbrace{\lim_{n\to\infty}\arctan(n+1)}^{\pi/2}}{\underbrace{\arctan 1}_{=\pi/4}}\right] = \ln \frac{\pi/2}{\pi/4}$$

$$= \ln \frac{4}{2} = \ln 2$$

$$\boxed{\lim_{x\to\infty} \arctan x = \frac{\pi}{2}}$$

Convergence of a series

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges to the sum** s, and we write

$$\sum_{n=1}^{\infty} a_n = s,$$

if $\lim_{n\to\infty} s_n = s$, where s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

Thus, a *series* converges if and only if the *sequence* of its partial sums converges.

Similarly, a series is said to diverge to infinity, diverge to negative infinity, or simply diverge if its sequence of partial sums does so.

2. Let $a_n > 0$ and $a_n \le a_{2n} + a_{2n+1}$ for all n. Then show that $\sum_{n=1}^{\infty} a_n$ diverges.

Assume that $\sum_{n=1}^{\infty} a_n$ converges then its partial sums converges

(ie $\lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} s_k = s$ for some $s \in \mathbb{R}$)

$a_1 \le a_2 + a_3$

$a_2 \le a_4 + a_5$

$a_3 \le a_6 + a_7$

$\vdots$

$a_n \le a_{2n} + a_{2n+1}$

$+$

$a_1 + a_2 + a_3 + \cdots + a_n \le a_2 + a_3 + \cdots + a_{2n+1}$ $+ a_1 - a_1$

$\sum_{k=1}^{n} a_k \le \sum_{k=1}^{2n+1} a_k - a_1$

$s_n \le s_{2n+1} - a_1$

$\lim_{n\to\infty} s_n \le \lim_{n\to\infty} s_{2n+1} - a_1$

$s \le s - a_1$

$\Rightarrow 0 \le -a$

$\Rightarrow a_1 \le 0$ # it's a contradiction

Therefore, $\sum_{n=1}^{\infty} a_n$ diverges

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Therefore, if $\lim_{n\to\infty} a_n$ does not exist, or exists but is not zero, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. (This amounts to an ***n*th term test for divergence of a series.**

3. Show that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \cos(a_n)$ diverges.

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \lim_{n\to\infty} \cos(a_n) = \cos(0) = 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} \cos(a_n)$ diverges

4. Find the sum of the following series.

$|r| < 1$

$1 + r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^{n-1}$

$= \frac{1}{1-r}$

(a) $\sum_{n=4}^{\infty} (-\frac{5}{8})^n$.

$\Rightarrow \left|-\frac{5}{8}\right| < 1$ so its series converges

$$\sum_{n=4}^{\infty}\left(-\frac{5}{8}\right)^n = \left(-\frac{5}{8}\right)^4 + \left(-\frac{5}{8}\right)^5 + \cdots$$

$$= \left(-\frac{5}{8}\right)^4 \left[1 + \left(-\frac{5}{8}\right)^1 + \left(-\frac{5}{8}\right)^2 + \cdots\right] =$$

$$= \left(-\frac{5}{8}\right)^4 \underbrace{\sum_{n=0}^{\infty}\left(-\frac{5}{8}\right)^n}_{\frac{1}{1-(-\frac{5}{8})} = \frac{8}{13}} = \frac{5^4}{8^4} \cdot \frac{8}{13} = \frac{5^4}{8^3 \cdot 13}$$

(b) $\sum_{n=1}^{\infty} \sin^{2n}\theta$, for $0 < \theta < \pi/2$.

$(\sin^2\theta)^n \longrightarrow (\sin^2\theta)^1 + (\sin^2\theta)^2 + \cdots$

$|\sin^2\theta| < 1$ for $0 < \theta < \pi/2$

$0 < \quad < 1$

So it converges and

$$\sum_{n=1}^{\infty}(\sin^2\theta)^n = \sin^2\theta \left(\underbrace{\sum_{n=0}^{\infty}(\sin^2\theta)^n}_{\frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}}\right) = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

The integral test

Suppose that $a_n = f(n)$, where f is positive, continuous, and nonincreasing on an interval $[N, \infty)$ for some positive integer N. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(t)\,dt$$

either both converge or both diverge to infinity.

$$\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges to infinity if } p \le 1. \end{cases}$$

P test

A comparison test

Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a positive constant K such that, ultimately, $0 \le a_n \le K b_n$.

(a) If the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

(b) If the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity, then so does the series $\sum_{n=1}^{\infty} b_n$.

A limit comparison test

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is either a nonnegative finite number or $+\infty$.

(a) If $L < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(b) If $L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges to infinity, then so does $\sum_{n=1}^{\infty} a_n$.

The ratio test

Suppose that $a_n > 0$ (ultimately) and that $\rho = \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists or is $+\infty$.

(a) If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \rho \le \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\rho = 1$, this test gives no information; the series may either converge or diverge to infinity.

The root test

Suppose that $a_n > 0$ (ultimately) and that $\sigma = \lim_{n\to\infty} (a_n)^{1/n}$ exists or is $+\infty$.

(a) If $0 \le \sigma < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \sigma \le \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\sigma = 1$, this test gives no information; the series may either converge or diverge to infinity.

5. Test the following series for convergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 - n}$.

We use the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{2n^2-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2n^2-n} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(2-\cancel{\frac{1}{n}}\right)} = \frac{1}{2}$$

$\frac{1}{n} \to 0$

and $0 \leq \frac{1}{2} < \infty$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-n}$ converges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$.

Harmonic series

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$

diverges to infinity

Since $0 < \ln k \leq k$ for $k \geq 2$

$\frac{1}{\ln k} \geq \frac{1}{k}$ and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges

So $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges by comparison

(c) $\sum_{k=1}^{\infty} \sin(1/k)$.

We use the LCT with $\sum_{k=1}^{\infty} \frac{1}{k}$

$\rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity (harmonic series)

$$\rightarrow \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{(\frac{0}{0})}{\underset{LH}{=}} \lim_{x\to\infty} \frac{\cos(\frac{1}{x})\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$$

$k \in \mathbb{N}$ $\quad x \in \mathbb{R}$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$$

$$= \cos(0) = 1$$

$0 < 1$

$\Rightarrow \sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ diverges by LCT with $\sum_{k=1}^{\infty} \frac{1}{k}$

(d) $\sum_{k=1}^{\infty} (\pi/2 - \arctan k^2)$.

We use the LCT with $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$\rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

$$\rightarrow \lim_{k\to\infty} \frac{\pi/2 - \arctan k^2}{\frac{1}{k^2}} = \lim_{x\to\infty} \frac{\pi/2 - \arctan x^2}{\frac{1}{x^2}} \overset{(\frac{0}{0})}{\underset{LH}{=}} \lim_{x\to\infty} \frac{+\frac{\cancel{2}x}{1+x^4}}{\frac{+\cancel{2}}{x^3}}$$

$k \in \mathbb{N}$ $\quad x \in \mathbb{R}$

$$= \lim_{x\to\infty} \frac{x^4}{1+x^4} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^4}} = 1$$

$0 \le 1 < \infty$

So, $\sum_{k=1}^{\infty} \pi/2 - \arctan k^2$ converges by the LCT with $\sum_{k=1}^{\infty} \frac{1}{k^2}$

6. Test the following series for convergence

(a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}}$.

$\frac{1}{n\sqrt{n^2+3}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n \cdot n} = \frac{1}{n^2} \Rightarrow \frac{1}{n\sqrt{n^2+3}} < \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges then

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}}$ converges by comparison

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(b) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$.

Let $a_n = \frac{n^3}{3^n}$ then $a_n > 0$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{\cancel{3^{n+1}}_{\,3}} \cdot \frac{\cancel{3^n}}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\underbrace{\cancel{\tfrac{1}{n}}}_{0}\right)^3 = \frac{1}{3}$$

and $0 \leq \frac{1}{3} < 1$

So, $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges by the ratio test

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$.

Let $a_n = \frac{1}{n^n}$ then $a_n > 0$ and

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$$ and

$0 \leq 0 < 1$

then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the root test

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(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Let $a_n = \frac{n!}{n^n}$ then $a_n > 0$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n\cancel{+1}}} \frac{n^n}{\cancel{n!}}$$

$$\boxed{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \to 2,\ldots$$

and $0 \leq \frac{1}{e} < 1$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the ratio test

Exercise!

(e) $\displaystyle\sum_{n=120}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}.$

$$(\ln(n))^{\ln(n)} \underset{x = e^{\ln x}}{=} e^{\ln[(\ln(n))^{\ln(n)}]}$$

$$= \left(e^{\ln(n)}\right)^{\ln[\ln(n)]} = n^{\ln[\ln(n)]}$$

$$n \geq 120 \Rightarrow \ln(\ln(n)) \geq \underbrace{\ln(\ln(120))}_{\cong 1,56} > 1,5$$

$$\Rightarrow \ln(\ln(n)) > 1,5 \Rightarrow n^{\ln(\ln(n))} > n^{1,5}$$

$$\Rightarrow \frac{1}{n^{\ln(\ln(n))}} < \frac{1}{n^{1,5}}$$

$\rightarrow \displaystyle\sum_{n=120}^{\infty} \frac{1}{n^{1,5}}$ converges by p-test $(1,5 > 1)$

$\Rightarrow \displaystyle\sum_{n=120}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges by comparison