Math 120 (Spring 2021)

Sections 71-72

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Recitation week 1

Def: (Page 439)

Limit of a sequence

We say that sequence $\{a_n\}$ converges to the limit L, and we write $\lim_{n\to\infty} a_n = L$, if for every positive real number ϵ there exists an integer N (which may depend on ϵ) such that if $n \geq N$, then $|a_n - L| < \epsilon$.

Theorem: (439)

If $\lim\limits_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim\limits_{n\to\infty} a_n = L$.

(when n is an integer)

If $\{a_n\}$ and $\{b_n\}$ converge then →

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n,$$

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{assuming } \lim_{n\to\infty} b_n \neq 0.$$

If $a_n \leq b_n$ ultimately, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

If $a_n \leq b_n \leq c_n$ ultimately, and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$.

Squeeze Theorem for Sequences:

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$

Thm: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

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1. Find the limit of the sequence $\{a_n\}$, where

(a) $a_n = \dfrac{\cos(4n)}{n+2}$ **(b)** $a_n = n - \sqrt{n^2 - 4n}$ **(c)** $a_n = \left(\dfrac{n+2}{n+1}\right)^n$ **(d)** $a_n = \dfrac{n!}{n^n}$

(a) $a_n = \dfrac{\cos(4n)}{n+2}$

$n \geq 1$

Since $-1 \leq \cos(4n) \leq 1 \Rightarrow \dfrac{-1}{n+2} \leq \dfrac{\cos(4n)}{n+2} \leq \dfrac{1}{n+2}$

$\lim\limits_{n\to\infty} \dfrac{-1}{n+2} = \lim\limits_{n\to\infty} \dfrac{1}{n+2} = 0$ then by squeeze thm

$\lim\limits_{n\to\infty} \dfrac{\cos(4n)}{n+2} = 0$

(b) $a_n = n - \sqrt{n^2 - 4n}$

$$\lim_{n\to\infty} n - \sqrt{n^2-4n} = \lim_{n\to\infty} \frac{\overbrace{(n+\sqrt{n^2-4n})(n-\sqrt{n^2-4n})}^{n^2 - (n^2-4n) = +4n}}{(n+\sqrt{n^2-4n})}$$

$$= \lim_{n\to\infty} \frac{4n}{n+\underbrace{\sqrt{n^2-4n}}_{\sqrt{n^2(1-\frac{4}{n})},\ |n| = n}} = \lim_{n\to\infty} \frac{4\cancel{n}}{\cancel{n}\left(1+\sqrt{1-\underbrace{\cancel{\tfrac{4}{n}}}_{0}}\right)}$$

$$= \frac{4}{2} = 2$$

(c) $a_n = \left(\frac{n+2}{n+1}\right)^n$

Let $f(x) = \left(1+\frac{1}{x+1}\right)^x$ and $f(n) = a_n$, $n \in \mathbb{N}$

$$\boxed{f(x) = e^{\ln f(x)}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(1+\frac{1}{x+1}\right)^x = e^{\lim_{x\to\infty}\left(\ln\left(\left(1+\frac{1}{x+1}\right)^x\right)\right)}$$

$$= e^{\lim_{x\to\infty}\left(x\ln\left(1+\frac{1}{x+1}\right)\right)} = e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x+1}\right)}{1/x}}$$

L'H

$$= e^{\lim_{x\to\infty} \frac{\frac{-1/(x+1)^2}{(x+2)/(x+1)}}{-1/x^2}} = e^{\lim_{x\to\infty} \frac{\cancel{-}1}{(x+1)^{\cancel{2}}} \cdot \frac{\cancel{x+1}}{x+2} \cdot \frac{x^2}{\cancel{-1}}}$$

$$= e^{\lim_{x\to\infty} \frac{x^2}{x^2+3x+2}} = e^{\lim_{x\to\infty} \frac{1}{1+\underbrace{\cancel{\frac{3}{x}}}_{0}+\underbrace{\cancel{\frac{2}{x^2}}}_{0}}} = e^1 = \lim_{x\to\infty} f(x)$$

$$\Rightarrow \lim_{n\to\infty} a_n = e \quad //$$

(d) $a_n = \dfrac{n!}{n^n}$

$$0 \le \underbrace{\frac{1}{n}}_{\le 1} \cdot \underbrace{\frac{2}{n}}_{\le 1} \cdot \frac{3}{n} \cdots \underbrace{\frac{n-1}{n}}_{\le 1} \cdot \underbrace{\frac{n}{n}}_{\le 1} \le \frac{1}{n}$$

$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$ then by squeeze thm

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \quad //$$

Thm: Suppose $\lim\limits_{n\to\infty} a_{2n} = L_1$ and $\lim\limits_{n\to\infty} a_{2n-1} = L_2$

if $L_1 = L_2$ then $\lim\limits_{n\to\infty} a_n = L_1$

if $L_1 \neq L_2$ then (a_n) is divergent

2. Let (a_n) be the sequence given as follows.

$$a_n = \begin{cases} \dfrac{2}{2n^3+n+1} & \text{if } n \text{ is even} \\[2ex] \dfrac{-3}{n^2+3n+5} & \text{if } n \text{ is odd} \end{cases}$$

If exists, find $\lim\limits_{n\to\infty} a_n$.

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2}{16n^3+2n+1} = \lim_{n\to\infty} \frac{2}{n^3\left(16+\underbrace{\cancel{\tfrac{2}{n^2}}}_{0}+\underbrace{\cancel{\tfrac{1}{n^3}}}_{0}\right)} = 0$$

$$\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \frac{-3}{4n^2+2n+3} = \lim_{n\to\infty} \frac{-3}{n^2\left(4+\underbrace{\cancel{\tfrac{2}{n}}}_{0}+\underbrace{\cancel{\tfrac{3}{n^2}}}_{0}\right)} = 0$$

$$\lim_{n\to\infty} a_n = 0$$

3. Find the limit of the sequence $\{a_n\}$, where $a_n = \sin(\pi n)$.

$$\{a_n\} = \{\sin\pi, \sin 2\pi, \ldots\} = \{0, 0, 0, \ldots\} \Rightarrow a_n = 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0 \;//$$

Bounded monotonic sequences converge

If the sequence $\{a_n\}$ is bounded above and is (ultimately) increasing, then it converges. The same conclusion holds if $\{a_n\}$ is bounded below and is (ultimately) decreasing.

4. Consider the sequence (a_n) defined recursively as follows.

$$a_1 = 1, \text{ and } a_n = 1 + \frac{a_{n-1}}{2} \text{ for all } n \geq 2$$

a) • Show that $a_n < 2$ for all n.

b) • Show that $a_n < a_{n+1}$ for all n.

c) • If exists, find the limit $\lim_{n\to\infty} a_n$.

a) → (it's true for $n=1$)

when $n=1$, $a_1 = 1 < 2$ ✓

→ (when it's true for $n=k$, it's also true $n=k+1$)

Assume that it is true for $n=k$

i.e $a_k < 2$

$\Rightarrow \frac{a_k}{2} < 1 \Rightarrow 1+\frac{a_k}{2} < 2$

$\Rightarrow a_{k+1} < 2$ ✓

by induction, $a_n < 2$ for all $n \in \mathbb{N}$

b) $a_n < a_{n+1}$ for all n

$\rightarrow$ for $n = 1$, $a_1 = 1$, $a_2 = 1 + \frac{a_1}{2} = \frac{3}{2}$

$\Rightarrow a_1 < a_2$ ✓

$\rightarrow$ Assume that it is true for $n = k$

ie $a_k < a_{k+1}$ $\Rightarrow \frac{a_k}{2} < \frac{a_{k+1}}{2}$

$\Rightarrow 1 + \frac{a_k}{2} < 1 + \frac{a_{k+1}}{2}$

$\Rightarrow a_{k+1} < a_{k+2}$

(it's true for $n = k+1$)

By induction, $a_n < a_{n+1}$ for all $n \geq 1$

c) Therefore, $\{a_n\}$ is bounded above and increasing then $\lim_{n\to\infty} a_n = L$ exists.

$$a_n = 1 + \frac{a_{n-1}}{2} \Rightarrow \underbrace{\lim_{n\to\infty} a_n}_{L} = \lim_{n\to\infty} \left(1 + \overbrace{\frac{a_{n-1}}{2}}^{L}\right)$$

$$\Rightarrow L = 1 + \frac{L}{2} \Rightarrow \frac{L}{2} = 1 \Rightarrow L = 2 //$$

5. Let (a_n) be the sequence defined recursively as follows.

$$a_1 = 1,\ a_2 = 2 \text{ and } \boxed{a_n = a_{n-1} + a_{n-2}} \text{ for all } n \geq 3$$

You are given the fact that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ has a positive limit. Find the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Let $c_n = \frac{a_{n+1}}{a_n}$ then $\boxed{c_1 = \frac{a_2}{a_1} = 2}$

$$c_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{c_n}$$

$\Rightarrow \boxed{c_{n+1} = 1 + \frac{1}{c_n}} \geq 1 > 0$

Let's $\lim_{n\to\infty} c_n = L$ (if c_n has a limit)

$$\underbrace{\lim_{n\to\infty} c_{n+1}}_{L} = \lim_{n\to\infty} 1 + \underbrace{\frac{1}{c_n}}_{L}$$

$$\Rightarrow L = 1 + \frac{1}{L} \Rightarrow L^2 = L + 1$$

$$\Rightarrow L^2 - L - 1 = 0$$

$$\Delta = 1 + 4 = 5 \Rightarrow L_{1,2} = \frac{1 \mp \sqrt{5}}{2}, \quad L \geq 0$$

$$\Rightarrow \underline{L = \frac{1 + \sqrt{5}}{2}} \quad \left(\text{golden ratio}\right)$$