Math 120 (Spring 2021)

Sections 21-22

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Recitation week 1

Def: (Page 439)

Limit of a sequence

We say that sequence $\{a_n\}$ converges to the limit L, and we write $\lim_{n\to\infty} a_n = L$, if for every positive real number ϵ there exists an integer N (which may depend on ϵ) such that if $n \geq N$, then $|a_n - L| < \epsilon$.

Theorem: (439)

$$\text{If } \lim_{x\to\infty} f(x) = L \text{ and } a_n = f(n), \text{ then } \lim_{n\to\infty} a_n = L.$$

(when n is an integer)

If $\{a_n\}$ and $\{b_n\}$ converge then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n,$$

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{assuming } \lim_{n\to\infty} b_n \neq 0.$$

If $a_n \leq b_n$ ultimately, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

If $a_n \leq b_n \leq c_n$ ultimately, and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Squeeze Theorem for Sequences:

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Thm: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

1. Find the limit of the sequence $\{a_n\}$, where

(a) $a_n = \dfrac{\cos(4n)}{n+2}$ **(b)** $a_n = n - \sqrt{n^2 - 4n}$ **(c)** $a_n = \left(\dfrac{n+2}{n+1}\right)^n$ **(d)** $a_n = \dfrac{n!}{n^n}$

(a) $a_n = \dfrac{\cos(4n)}{n+2}$ $n \geq 1$

Since $-1 \leq \cos(4n) \leq 1 \Rightarrow \dfrac{-1}{n+2} \leq \dfrac{\cos(4n)}{n+2} \leq \dfrac{1}{n+2}$

$\lim_{n\to\infty} \dfrac{-1}{n+2} = \lim_{n\to\infty} \dfrac{1}{n+2} = 0$ then by squeeze thm

$\lim_{n\to\infty} \dfrac{\cos(4n)}{n+2} = 0$

(b) $a_n = n-\sqrt{n^2-4n}$

$$\underset{\substack{\| \\ n^2 - (n^2-4n) \\ \| \\ +4n}}{}$$

$$\lim_{n\to\infty} n-\sqrt{n^2-4n} = \lim_{n\to\infty} \frac{(n+\sqrt{n^2-4n})(n-\sqrt{n^2-4n})}{(n+\sqrt{n^2-4n})}$$

$$= \lim_{n\to\infty} \frac{4n}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{4\cancel{n}}{\cancel{n}\left(1+\sqrt{1-\cancel{\frac{4}{n}}}\right)}$$

$$\sqrt{n^2\left(1-\frac{4}{n}\right)} \qquad |n| = n \qquad \frac{4}{n}\to 0$$

$$= \frac{4}{2} = 2$$

(c) $a_n = \left(\dfrac{n+2}{n+1}\right)^n$

$$f(x) = \left(\frac{x+2}{x+1}\right)^x = \left(1+\frac{1}{x+1}\right)^x$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^x = e^{\lim_{x\to\infty} \ln\left(1+\frac{1}{x+1}\right)^x}$$

$$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x+1}\right)} = e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x+1}\right)}{\frac{1}{x}}}$$

L'H

$$= e^{\lim_{x\to\infty} \frac{\frac{-1}{(x+1)^2} / \left(1+\frac{1}{x+1}\right)}{-1/x^2}} \qquad \left(1+\frac{1}{x+1}\right) \to \frac{x+2}{x+1}$$

$$= e^{\lim_{x\to\infty} \frac{\cancel{-1}}{(x+1)^{\cancel{2}}}\cdot\frac{\cancel{x+1}}{x+2}\cdot\frac{x^2}{\cancel{-1}}}$$

$$= e^{\lim_{x\to\infty} \frac{x^2}{x^2+3x+2}} = e^{\lim_{x\to\infty} \frac{1}{1+\cancel{\frac{3}{x}}+\cancel{\frac{2}{x^2}}}} = e^1$$

$$\frac{3}{x}\to 0 \qquad \frac{2}{x^2}\to 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = e \quad //$$

(d) $a_n = \dfrac{n!}{n^n}$

$$0 \leq \frac{1}{n} \cdot \underbrace{\frac{2}{n}}_{\leq 1} \cdot \underbrace{\frac{3}{n}}_{\leq 1} \cdots \underbrace{\frac{n-1}{n}}_{\leq 1} \cdot \underbrace{\frac{n}{n}}_{\leq 1} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{(by squeeze thm)}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Thm: Suppose $\lim_{n\to\infty} a_{2n} = L_1$ and $\lim_{n\to\infty} a_{2n-1} = L_2$

if $L_1 = L_2$ then $\lim_{n\to\infty} a_n = L_1$

if $L_1 \neq L_2$ then (a_n) is divergent

2. Let (a_n) be the sequence given as follows.

$$a_n = \begin{cases} \dfrac{2}{2n^3+n+1} & \text{if } n \text{ is even} \\[2ex] \dfrac{-3}{n^2+3n+5} & \text{if } n \text{ is odd} \end{cases}$$

If exists, find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2}{16n^3+2n+1} = \lim_{n\to\infty} \frac{2}{n^3\left(16+\cancel{\frac{2}{n^2}}+\cancel{\frac{1}{n^3}}\right)} = 0$$

$$\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \frac{-3}{(2n-1)^2+3(2n-1)+5} = \lim_{n\to\infty} \frac{-3}{4n^2+2n+3}$$

$$= \lim_{n\to\infty} \frac{-3}{n^2\left(4+\cancel{\frac{2}{n}}+\cancel{\frac{3}{n^2}}\right)} = 0$$

(by thm)

$\Rightarrow \lim_{n\to\infty} a_n = 0$

3. Find the limit of the sequence $\{a_n\}$, where $a_n = \sin(\pi n)$.

$\{a_n\} = \{\underline{\sin \pi}, \underline{\sin 2\pi}, \underline{\sin 3\pi}, \dots\} = \{\underline{0}, \underline{0}, \underline{0}, \dots\}$

$a_n = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ //

Bounded monotonic sequences converge

If the sequence $\{a_n\}$ is bounded above and is (ultimately) increasing, then it converges. The same conclusion holds if $\{a_n\}$ is bounded below and is (ultimately) decreasing.

4. Consider the sequence (a_n) defined recursively as follows.

$$a_1 = 1, \text{ and } a_n = 1 + \frac{a_{n-1}}{2} \text{ for all } n \geq 2$$

a) • Show that $a_n < 2$ for all n.

b) • Show that $a_n < a_{n+1}$ for all n.

c) • If exists, find the limit $\lim_{n\to\infty} a_n$.

a) From the indiction,

→ $a_1 = 1 < 2$ ✓

→ Assume that it is true for $n = k$

so $a_k < 2$

then $\frac{a_k}{2} < 1 \Rightarrow 1 + \frac{a_k}{2} < 2 \Rightarrow a_{k+1} < 2$ ✓

by induction, $a_n < 2$ for all $n \geq 1$

b) $a_n < a_{n+1}$ for all $n \geq 1$ (?)

$\rightarrow$ for $n = 1$

$a_1 = 1, \quad a_2 = 1 + \frac{a_1}{2} = 1 + \frac{1}{2} = \frac{3}{2} \Rightarrow a_1 < a_2$ ✓

$\rightarrow$ Assume that it is true for $n = k$

So $a_k < a_{k+1} \Rightarrow \frac{a_k}{2} < \frac{a_{k+1}}{2}$

$\Rightarrow 1 + \frac{a_k}{2} < 1 + \frac{a_{k+1}}{2}$

$\Rightarrow a_{k+1} < a_{k+2}$ ✓

(true for $n = k+1$)

by induction, $a_n < a_{n+1}$ for all $n \geq 1$

c) Therefore, $\{a_n\}$ is bounded above and increasing then $\lim_{n \to \infty} a_n = L$ exists.

$$a_n = 1 + \frac{a_{n-1}}{2} \Rightarrow \underbrace{\lim_{n \to \infty} a_n}_{L} = \lim_{n \to \infty} 1 + \overbrace{\frac{a_{n-1}}{2}}^{L}$$

$$\Rightarrow \underline{L = 1 + \frac{L}{2}}$$

$$\Rightarrow \frac{L}{2} = 1 \Rightarrow L = 2 \,//$$

5. Let (a_n) be the sequence defined recursively as follows.

$$a_1 = 1,\ a_2 = 2 \text{ and } a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 3$$

You are given the fact that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ has a positive limit. Find the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Let $c_n = \frac{a_{n+1}}{a_n}$. If we know that the limit of c_n exists then let's say

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

$$c_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{c_n}$$

Then, $\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} 1 + \frac{1}{c_n}$

$$\Rightarrow \quad L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0$$

$$\Delta = 1 + 4 \Rightarrow L_{1,2} = \frac{1 \mp \sqrt{5}}{2}$$

c_n is positive then $L = \frac{1+\sqrt{5}}{2}$ //