MATH119 - Rec. - Week 13

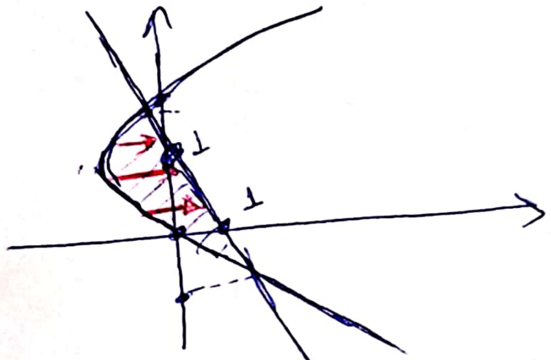
① Find the area of the indicated region

(a) bounded by the parabola $x=(y-1)^2-1$ and the line $x+y=1$.

(b) " " " $y=2x^2$ and $y=-x^2+3x+6$

Solution:

a)



$$\underbrace{x=(y-1)^2-1}$$

$x=0 \Rightarrow y=0$

$y=2$

$y=0 \Rightarrow x=0$

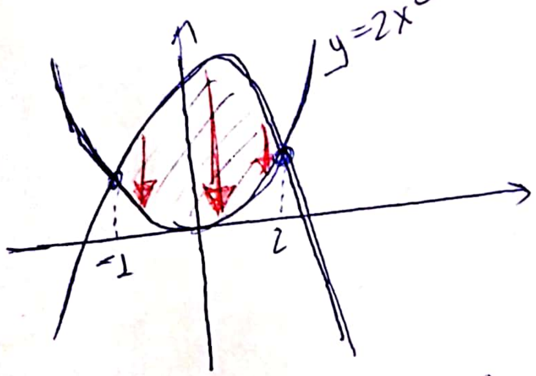
$y=1-x \Rightarrow x=(1-x-1)^2-1 \Rightarrow x^2-x-1=0$, $\Delta = 1-4\cdot1\cdot(-1)=5$

$$x_{1,2}=\frac{1\pm\sqrt{5}}{2}=\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

$$\text{Area} = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \left[(1-y)-\left((y-1)^2-1\right)\right]dy$$

$x_1 = \frac{1-\sqrt{5}}{2} \Rightarrow y_1 = 1-\frac{1-\sqrt{5}}{2}=\frac{1+\sqrt{5}}{2}$, $x_2=\frac{1+\sqrt{5}}{2} \Rightarrow y_2 = 1-\frac{1+\sqrt{5}}{2}=\frac{1-\sqrt{5}}{2}$

b)



$y=2x^2$, $\underbrace{y=-x^2+3x+6}$

$\Delta = 9+4\cdot6=33$

$$x_{1,2}=\frac{-3\pm\sqrt{33}}{-2}$$

$$\text{Area} = \int_{-1}^{2}\left((-x^2+3x+6)-(2x^2)\right)dx$$

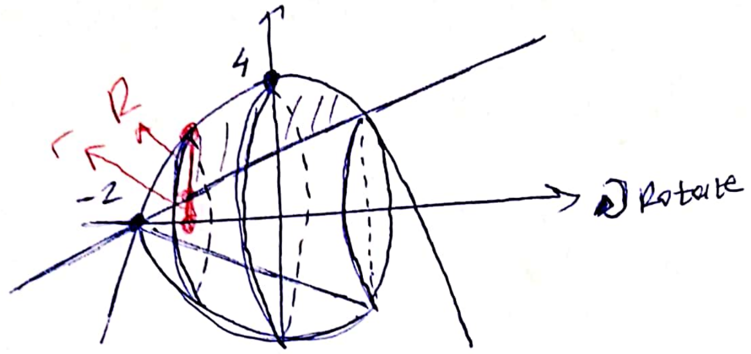
$y=2x^2 \Rightarrow 2x^2=-x^2+3x+6 \Rightarrow 3x^2-3x-6=0 \Rightarrow x^2-x-2=0$

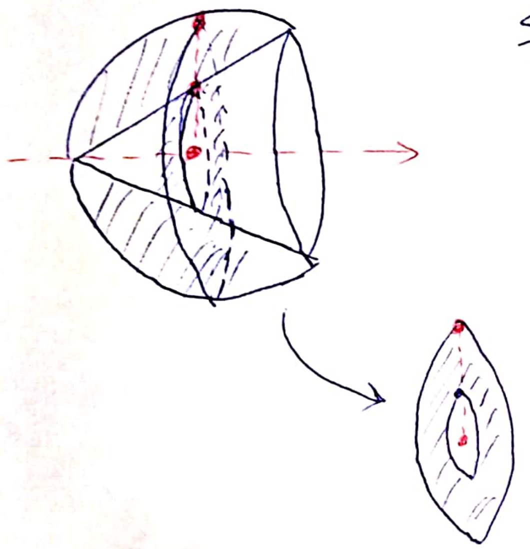
$\Rightarrow (x-2)(x+1)=0$

② Let $f(x) = 4 - x^2$ and $g(x) = 2 + x$. Find the volume of a solid S obtained by rotating the region enclosed by f and g about the x-axis. (Do not Evaluate)

$y = 4 - x^2, \quad y = 2 + x$

Solution:





Slicing (Disk) Method:

$$\text{Volume} = \int \pi \cdot (R^2 - r^2)\,dx$$

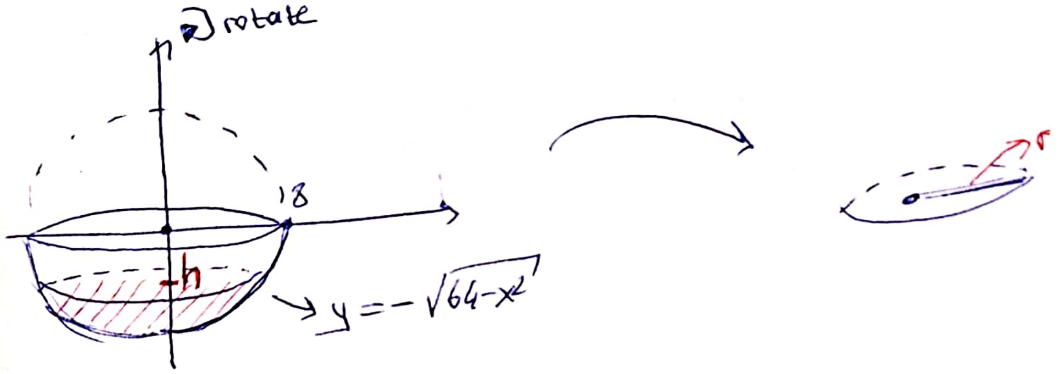
$$= \int_{-2}^{1} \pi \cdot \left((4 - x^2)^2 - (2 + x)^2 \right) dx$$

$4 - x^2 = 2 + x \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x + 2)(x - 1) = 0$

(3) Express the volume $V(h)$ of water in a 16cm-diameter hemispherical bowl as an integral and, hence, as a func. of the depth h of the water.

Solution: Since the bowl is hemispherical, it can be generated by rotating the quarter-disk, $-\sqrt{64-x^2} \leq y \leq 0$, $0 \leq x \leq 8$, about the y-axis.



Let us use Washer (Slicing) Method:

$$\text{Volume} = \int \pi \cdot r^2 \, dy = \int_{-8}^{h} \pi \cdot \left(\sqrt{64-y^2}\right)^2 dy$$

$$\left(y = -\sqrt{64-x^2} \Rightarrow x^2+y^2=64 \Rightarrow x = \pm\sqrt{64-y^2} \right)$$

$$= \pi \cdot \int_{-8}^{h} (64-y^2)\, dy = \pi \cdot \left(64y - \frac{y^3}{3} \Big|_{-8}^{h} \right)$$

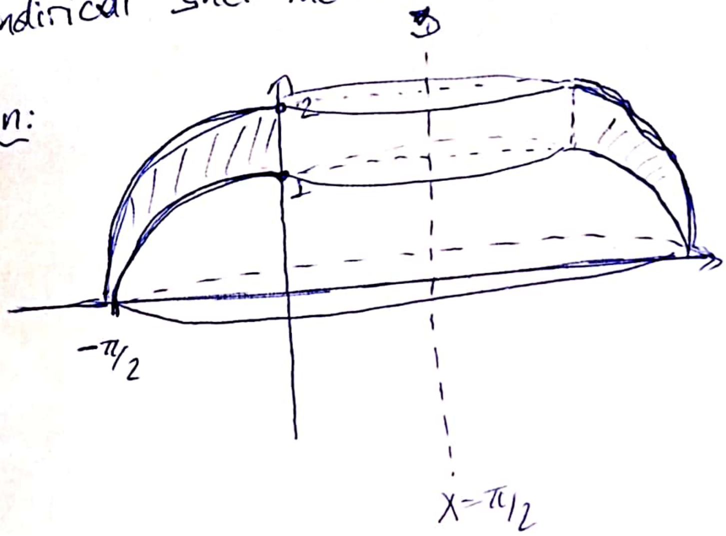
$$= \pi \cdot \left(64h - \frac{h^3}{3} - \left(64 \cdot (-8) + \frac{8^3}{3} \right) \right)$$

(4) Let S be the solid obtained by rotating the finite plane region bounded by the curves $y = 2\cos x$, $y = \cos x$, $x = 0$ and $x = -\pi/2$ about the line $x = \pi/2$. Write down the integrals (without evaluating) giving the volume of S by using

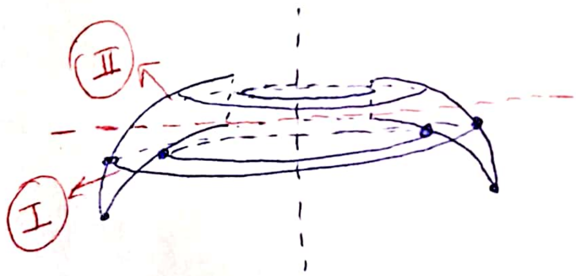
(a) slicing (disk) method

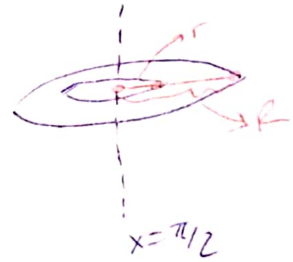
(b) Cylindirical shell method

Solution:



a)





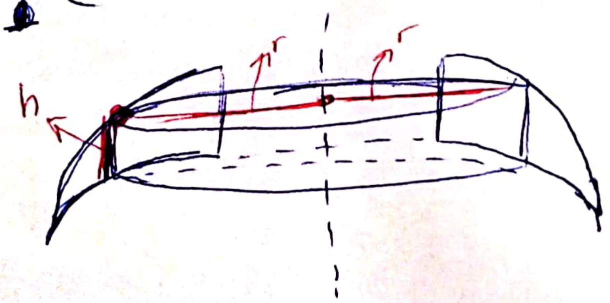
I $\begin{cases} R = \arccos(y/2) + \pi/2 \\ r = \arccos y + \pi/2 \end{cases}$

II $\begin{cases} R = \arccos(y/2) + \pi/2 \\ r = \pi/2 \end{cases}$

$$Vol = \int \pi \cdot (R^2 - r^2)\,dy = \pi \cdot \int_0^1 \left(\left(\arccos(y/2) + \pi/2\right)^2 - \left(\arccos y + \pi/2\right)^2 \right) dy$$

$$+ \pi \cdot \int_1^2 \left(\left(\arccos(y/2) + \pi/2\right)^2 - (\pi/2)^2 \right) dy$$

b)



$$Vol = \int 2\pi r \cdot h\,dx$$

$$= 2\pi \int_{-\pi/2}^{0} \left(\frac{\pi}{2} + x\right) \cdot (2\cos x - \cos x)\,dx$$