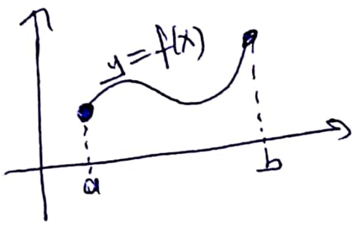
MATH119 – Recit – Week 14

① Find the length of the curve $y = \ln(\sec x)$ for $0 \le x \le \pi/4$.

Solution: Recall that the arc length formula for a smooth curve $y = f(x)$ from $x = a$ to $x = b$ is

$$s = \int_a^b \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{=ds} \quad \text{where}$$



$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ is called the "arclength element".

Here, $\frac{dy}{dx} = \sec x \cdot \tan x \cdot \frac{1}{\sec x} = \tan x$ is cont. on $[0, \pi/4]$,

so $y = \ln(\sec x)$ is a smooth curve on $[0, \pi/4]$.

$$\text{The arc length} = s = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln|\sqrt{2}+1| - \ln|1|$$

$$= \ln(\sqrt{2}+1)$$

$\sec x \geq 0$ on $[0, \pi/4]$

(2) Set up, but do NOT evaluate, integrals for the lengths of the curves

(a) $y = 2^x$, $0 \le x \le 3$

(b) $y = x - y^3$, $1 \le y \le 4$

Solution: a) $\frac{dy}{dx} = 2^x \cdot \ln 2$ is cont. ~~for all~~ $x \in [0,3]$, so the curve is smooth for all $x \in [0,3]$. The length s is

$$s = \int_{x=0}^{x=3} ds = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + 2^{2x} \cdot (\ln 2)^2}\, dx$$

b) It will be easier if we consider the equation of the curve as $x = y + y^3$, $1 \le y \le 4$.

$\frac{dx}{dy} = 1 + 3y^2$ is cont. for all $1 \le y \le 4$, so it is smooth.

The length $$s = \int_{y=1}^{y=4} ds = \int_{y=1}^{y=4} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^4 \sqrt{1 + (1 + 3y^2)^2}\, dy$$

(3) Find the area of the surface obtained by rotating the curve about the x-axis

(a) $9x = y^2 + 18$, $2 \le x \le 6$

(b) $y = \cos 2x$, $0 \le x \le \pi/6$

(c) $x = 1 + 2y^2$, $1 \le y \le 2$

(d) $y = e^{-x}$, $x \ge 0$

Solution: If $f'(x)$ is cont. on $[a,b]$ and $y=f(x)$ is rotated about the x-axis, then the area of the surface of revolution is

$$S = 2\pi \cdot \int_{x=a}^{x=b} |y| . ds = 2\pi . \int_a^b |f(x)| . \sqrt{1+(f'(x))^2}\, dx$$

If $g'(y)$ is cont. on $[c,d]$ and $x=g(y)$ is rotated about the x-axis,

$$S = 2\pi \cdot \int_{y=c}^{y=d} |y| ds = 2\pi \int_c^d |y| . \sqrt{1+(g'(y))^2}\, dy$$

a) $9x = y^2+18 \Rightarrow 9\frac{dx}{dy} = 2y \Rightarrow \frac{dx}{dy} = \frac{2}{9}y$ is cont. on $[0,6]$

$\begin{pmatrix} x=2 \Rightarrow y=0 \\ x=6 \Rightarrow y=\pm 6 \end{pmatrix}$ $\quad S = 2\pi \cdot \int_0^6 |y| . \sqrt{1+\left(\frac{2y}{9}\right)^2}\, dy = 2\pi \int_0^6 y . \sqrt{81+4y^2} . \frac{1}{9} dy$

b) $y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x$ is cont. on $[0, \pi/6]$. So, $y=\cos 2x$ is smooth.

$$S = 2\pi \int_0^{\pi/6} |y| . \sqrt{1+(-2\sin 2x)^2}\, dx = \int_0^{\pi/6} 2\pi . \cos(2x) . \sqrt{1+4\sin^2(2x)}\, dx$$

$$= 2\pi . \int_0^{\sqrt{3}} \frac{1}{4} . \sqrt{1+u^2}\, du = \frac{\pi}{2} . \underbrace{\int_0^{\pi/3} \sec\theta . \sec^2\theta\, d\theta}_{I} =$$

$\boxed{2\sin 2x = u \Rightarrow 4\cos 2x dx = du}$ $\quad \boxed{u = \tan\theta \Rightarrow du = \sec^2\theta d\theta}$ $\quad \boxed{\begin{array}{l} \sec\theta = u \Rightarrow \sec\theta \tan\theta d\theta = du \\ \sec^2\theta d\theta = dv \Rightarrow \tan\theta = v \end{array}}$

$$= \frac{\pi}{2} . \left(\sec\theta . \tan\theta \Big|_0^{\pi/3} - \int_0^{\pi/3} \tan\theta . \sec\theta \tan\theta d\theta \right)$$

$$= \frac{\pi}{2}\left(\sec\theta \tan\theta \Big|_0^{\pi/3} - \left(\int_0^{\pi/3} \sec\theta (\sec^2\theta - 1) d\theta \right)\right) = \frac{\pi}{2} . \left(2\sqrt{3} + \int_0^{\pi/3} \sec\theta d\theta - \underbrace{\int_0^{\pi/3} \sec^3\theta d\theta}_{I}\right.$$

c) $x = 1 + 2y^2 \Rightarrow \frac{dx}{dy} = 4y$ is cont. on $[1,2]$, so the curve is smooth.

$$S = 2\pi \int_1^2 |y| \, ds = 2\pi \int_1^2 y \cdot \sqrt{1 + 16y^2} \, dy = \ldots$$

$$\boxed{\begin{array}{l} 1 + 16y^2 = u \\ 32y\,dy = du \end{array}}$$

d) $y = e^{-x}$, $x \geq 0 \Rightarrow \frac{dy}{dx} = -e^{-x}$ is cont. on $[0, \infty)$, so the curve is smooth.

$$S = 2\pi \cdot \int_{x=0}^{x=\infty} |y| \cdot ds = 2\pi \int_0^\infty e^{-x} \cdot \sqrt{1 + (-e^{-x})^2} \, dx = 2\pi \int_0^\infty e^{-x} \sqrt{1 + e^{-2x}} \, dx$$

$$= 2\pi \cdot \lim_{R \to \infty} \int_0^R e^{-x} \cdot \sqrt{1 + e^{-2x}} \, dx = -2\pi \cdot \lim_{R \to \infty} \int_1^{e^{-R}} \sqrt{1 + u^2} \, du$$

$$\boxed{\begin{array}{l} e^{-x} = u \\ -e^{-x} dx = du \end{array}}$$

$$= -2\pi \lim_{R \to \infty} \int_{\pi/4}^{\arctan(e^{-R})} \sec\theta \sec^2\theta \, d\theta$$

$$\boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta \, d\theta \end{array}}$$

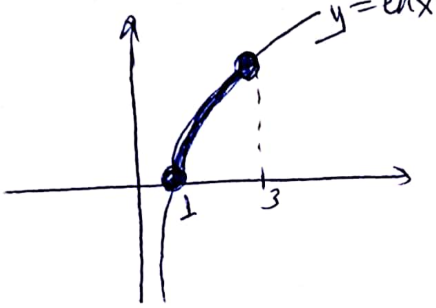
(4) Set up, but do NOT evaluate, integrals for the area of the surface obtained by rotating the curve about the given axis.

(a) $y = \ln x$, $1 \le x \le 3$; x-axis and y-axis

(b) $y = \sin^2 x$, $0 \le x \le \pi/2$; x-axis

(c) $y = \sec x$, $1 \le y \le \pi/4$; y-axis

Solution: a)



about x-axis

$$S = 2\pi \cdot \int_1^3 \ln x \cdot \sqrt{1 + \frac{1}{x^2}}\, dx$$

about y-axis

$$S = 2\pi \cdot \int_1^3 x \cdot \sqrt{1 + 1/x^2}\, dx$$

b) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = 2\sin x \cos x = \sin(2x)$ is cont. on $0 \le x \le \pi/2$,

so the curve is smooth.

$$S = \int_0^{\pi/2} 2\pi \cdot \sin^2 x \cdot \sqrt{1 + \sin^2(2x)}\, dx$$

c) $y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$ is cont. on $[0, \arccos(4/\pi)]$

$y = 1 \Rightarrow x = 0$

$y = \pi/4 \Rightarrow x = \arccos(4/\pi)$

$$S = \int_0^{\arccos(4/\pi)} 2\pi \cdot x \cdot \sqrt{1 + \sec^2 x \tan^2 x}\, dx$$

(5) Let C be the segment of the plane curves defined by

(i) $y=e^x$ between $y=1$ and $y=e$;

(ii) $y=e^{x/4}$ between $y=1$ and $y=e$.

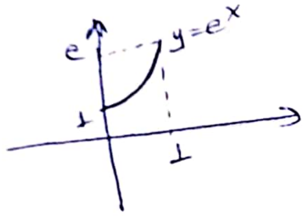
(a) Set up, but do NOT evaluate, an integral wrt y for the arclength of C

(b) " " " x "

(c) " , " for the area of the surface obtained by rotating C about the y-axis.

(d) Set up, but do NOT evaluate, an integral for the area of the surface obtained by rotating C about the x-axis.

Solution: (i)



a) $s = \int_1^e \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_1^e \sqrt{1+ 1/y^2}\, dy$

$y=e^x \Rightarrow x=\ln y \Rightarrow \frac{dx}{dy}=\frac{1}{y}$

b) $s = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+e^{2x}}\, dx$

c) $S = 2\pi \cdot \int_0^1 x \cdot \sqrt{1+e^{2x}}\, dx$

d) $S = 2\pi \cdot \int_1^e y \cdot \sqrt{1+1/y^2}\, dy$

(6) Set up, but do not evaluate, for the areas of the surfaces generated by revolving the curves (i) $y=\sqrt{x}$, $1 \le x \le 9$,

(ii) $y = \sin x$, $0 \le x \le \pi/2$

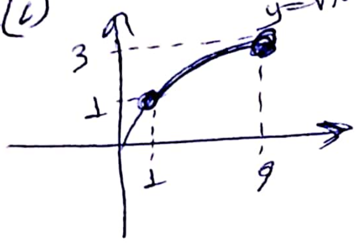
(a) around x-axis in dx form

(b) " x-axis in dy form

(c) " y-axis in dx form

(d) " y-axis in dy form

Solution: (i) $y=\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $x=y^2 \Rightarrow \frac{dx}{dy} = 2y$



a) $$S = \int_1^9 2\pi \cdot \sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}}\, dx$$

b) $$S = \int_1^3 2\pi \cdot y \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^3 2\pi \cdot y \cdot \sqrt{1+4y^2}\, dy$$

c) $$S = \int_1^9 2\pi \cdot x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^9 2\pi x \cdot \sqrt{1 + \frac{1}{4x}}\, dx$$

d) $$S = \int_1^3 2\pi \cdot y^2 \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^3 2\pi y^2 \cdot \sqrt{1+4y^2}\, dy$$