MATH119 - Recit. - Week 11

① Evaluate the indefinite integral

(a) $\int x^2 . e^{4+x^3} . dx$ (b) $\int \frac{x - \sin x}{x^2 + 2\cos x} dx$ (c) $\int \frac{x^2+1}{\sqrt{x^3+3x-2}} dx$

(d) $\int 3^x . dx$ (e) $\int e^x . \sqrt{1+4e^x} dx$ (f) $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$

Solution: a) $\int x^2 . e^{4+x^3} . dx = \int \frac{1}{3} . e^u . du = \frac{e^u}{3} + c = \frac{e^{4+x^3}}{3} + c$

$\boxed{4+x^3 = u \Rightarrow 3x^2 dx = du}$

c) $\int \frac{x^2+1}{\sqrt{x^3+3x-2}} dx = \frac{1}{3} . \int \frac{du}{\sqrt{u}} = \frac{1}{3} . \int u^{-1/2} . du = \frac{2}{3} . u^{1/2} + c$

$= \frac{2}{3} . \sqrt{x^3+3x-2} + c$

$\boxed{x^3+3x-2 = u \Rightarrow (3x^2+3) dx = du}$

(f) $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx = 2 . \int \sec^2 u . du = 2 . \tan u + c$

$= 2 . \tan(\sqrt{x}) + c$

$\boxed{\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}} dx = du}$

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② Evaluate the following integrals.

(a) $\int \sin^5 x . \cos x . dx$ (b) $\int \sin^2 x . \cos^2 x . dx$ (c) $\int \sec^6 x . \tan^2 x . dx$

(d) $\int \sec^3 x . \tan^3 x . dx$

Solution: b) $\int \sin^2 x . \cos^2 x dx = \int (\sin x . \cos x)^2 dx = \int \left(\frac{\sin 2x}{2}\right)^2 dx = \frac{1}{4} . \int \sin^2(2x) dx$

$= \frac{1}{4} . \int \left(\frac{1 - \cos(4x)}{2}\right) dx = \frac{1}{8} . \left(\int dx - \int \cos(4x) dx\right) = \frac{1}{8} . \left(x - \frac{\sin 4x}{4}\right) + c$

$\boxed{\cos 2x = 1 - 2\sin^2 x}$

d) $\int \sec^3 x \cdot \tan^3 x \, dx = \int \sec^2 x \cdot \tan^2 x \cdot \sec x \cdot \tan x \cdot dx$

$= \int u^2 \cdot (u^2 - 1) \cdot du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c.$

$\boxed{\sec x = u \Rightarrow \sec x \cdot \tan x \cdot dx = du \text{ and } 1 + \tan^2 x = \sec^2 x}$

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3) Evaluate the following integrals

(a) $\int (x+3) e^{2x} \cdot dx$ (b) $\int_1^e \sin(\ln x) dx$ (c) $\int x \sin^2 x \cdot dx$

Solution: Recall the formula for the method of integration by parts;

$$\int u \cdot dv = u \cdot v - \int v \cdot du \quad , \quad (\text{LAPTE})$$

a) $\int (x+3) e^{2x} dx = (x+3) \frac{e^{2x}}{2} - \frac{1}{2} \int e^{2x} \cdot dx = (x+3) \frac{e^{2x}}{2} - \frac{1}{4} e^{2x} + C$

$x+3 = u \Rightarrow dx = du$

$e^{2x} dx = dv \Rightarrow \frac{e^{2x}}{2} = v$

b) Let $I = \int \sin(\ln x) dx$.

$I = \int \sin(\ln x) dx = \sin(\ln x) \cdot x - \int \cos(\ln x) dx = \sin(\ln x) \cdot x - \left(\cos(\ln x) \cdot x + \int \sin(\ln x) dx \right)$

$\sin(\ln x) = u \Rightarrow \frac{\cos(\ln x)}{x} \cancel{dx} = du$

$dx = dv \Rightarrow x = v$

$\cos(\ln x) = u \Rightarrow -\frac{\sin(\ln x)}{x} dx = du$

$dx = dv \Rightarrow x = v$

So, we get $I = \sin(\ln x) \cdot x - \cos(\ln x) \cdot x - I \Rightarrow I = \frac{(\sin(\ln x) - \cos(\ln x)) \cdot x}{2}$

Then, $\int_1^e \sin(\ln x) dx = \frac{1}{2} \cdot (\sin(\ln x) - \cos(\ln x)) \cdot x \Big|_1^e = \frac{1}{2} \cdot e \cdot (\sin(1) - \cos(1)) - \frac{1}{2} \cdot (-1)$

4) Evaluate the following integrals.

(a) $\int \frac{x^2-8}{x^2-9}\,dx$ (b) $\int \frac{3x}{(x-1)^2\cdot(x^2+x+1)}\,dx$

Solution: b) By using partial fractions,

$$\frac{3x}{(x-1)^2\cdot(x^2+x+1)} = \frac{A}{\underset{(x-1)(x^2+x+1)}{x-1}} + \frac{B}{\underset{(x^2+x+1)}{(x-1)^2}} + \frac{Cx+D}{\underset{(x-1)^2}{x^2+x+1}}$$

$$\Rightarrow \left.\begin{array}{l} (A+C)\cdot x^3 = 0 \\ (B-2C+D)x^2 = 0 \\ (B+C-2D)\cdot x = 3x \\ -A+B+D = 0 \end{array}\right\} \Rightarrow \begin{array}{l} C=-A \\ B+D=A \\ A-2\underset{-A}{C}=0 \Rightarrow A=0=C \\ \Rightarrow D=-B, \quad B+C-2D=3B+C=3 \\ B=1 \\ D=-1 \end{array}$$

So, $\int \frac{3x}{(x-1)^2(x^2+x+1)}\,dx = \int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{x^2+x+1}\,dx$

$$\Rightarrow \int \frac{1}{(x-1)^2}\,dx = \int \frac{du}{u^2} = -\frac{1}{x-1} + C$$

$\boxed{x-1=u \Rightarrow dx=du}$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{dx}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2} = \int \frac{dx}{\frac{3}{4}\cdot\left(1+\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)^2\right)}$$

$$= \frac{4}{3}\cdot\int \frac{dx}{1+\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)^2} = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{du}{1+u^2} = \frac{2\sqrt{3}}{3}\cdot\arctan u + C$$

$$= \frac{2\sqrt{3}}{3}\cdot\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

$$\frac{x+\frac{1}{2}}{\sqrt{3}/2} = u \Rightarrow \frac{2}{\sqrt{3}}\cdot dx = du$$

So, $\int \frac{3x}{(x-1)^2(x^2+x+1)}\,dx = \frac{-1}{x-1} + \frac{2\sqrt{3}}{3}\cdot\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$

(5) Evaluate the following integrals

(a) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$ (b) $\displaystyle\int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx$

Solution: a) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{2\sec^2\theta\,d\theta}{(4\sec^2\theta)^2} = \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta}$

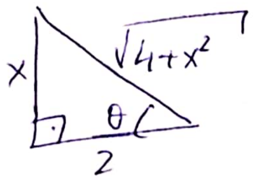
Use the inverse tangent substitution, i.e., $\boxed{x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta}$

$$= \frac{1}{8}\cdot\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\cdot\int \cos^2\theta\,d\theta = \frac{1}{8}\cdot\frac{1}{2}\cdot\int(\cos 2\theta + 1)\,d\theta$$

$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$

$$= \frac{1}{16}\cdot\left(\int \cos 2\theta\,d\theta + \int d\theta\right) = \frac{1}{16}\cdot\left(\frac{\sin 2\theta}{2} + \theta\right) + C,$$

(Now do the back-substitution, i.e., $\tan\theta = \frac{x}{2}$



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$$= \frac{1}{16}\cdot\left(\frac{2\sin\theta\cdot\cos\theta}{2} + \theta\right) + C = \frac{1}{16}\cdot\left(\frac{x}{\sqrt{4+x^2}}\cdot\frac{2}{\sqrt{4+x^2}} + \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$