MATH119 - Recit. - Week 10

① Interpret the given sum S_n as a sum of rectangles approximating the area of a certain region in the plane and compute $\lim_{n\to\infty} S_n$ where $S_n = \sum_{i=1}^{n} \frac{i + n\cdot\sqrt{1-(i/n)^2}}{n^2}$

Solution: Rewrite S_n as

$$\sum_{i=1}^{n} \frac{1}{n}\cdot\left(\frac{i}{n} + \sqrt{1-\left(\frac{i}{n}\right)^2}\right)$$

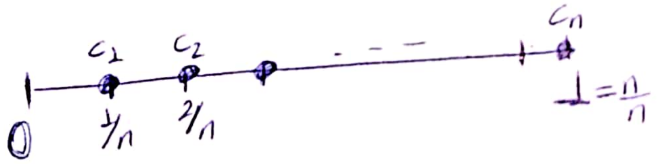
Recall the General Riemann Sums; $R(f,P) = \sum_{i=1}^{n} f(c_i)\cdot\Delta x_i$

where $c_i \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$. $1 \le i \le n$.

In this case, $\Delta x_i = \frac{1}{n}$, $c_i = \frac{i}{n}$, $f(x) = x + \sqrt{1-x^2}$.

Here since $\Delta x_i = \frac{1}{n}$ we should have an interval of length 1.

(Divided into n equal length subintervals).

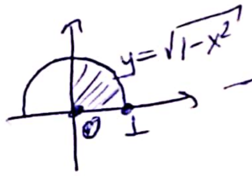


$[x_{i-1}, x_i] = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ where $1 \le i \le n$

$$\lim_{n\to\infty} S_n = \int_0^1 f(x)\,dx = \int_0^1 \left(x+\sqrt{1-x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 + \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2} + \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2} + \frac{\pi}{4}$$

$y = \sqrt{1-x^2} \Rightarrow x^2+y^2=1$

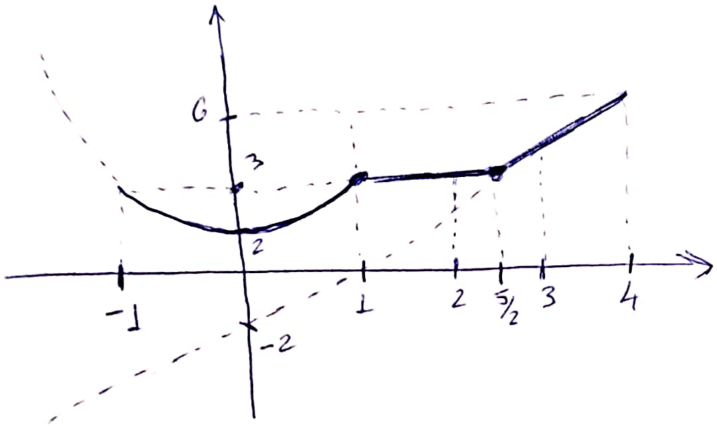


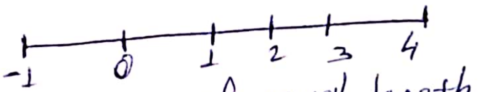
(2) Consider the function

$$f(x)=\begin{cases} x^2+2 & \text{if } x\le 1 \\ 3 & \text{if } 1<x\le 5/2 \\ 2x-2 & \text{if } x>5/2. \end{cases}$$

over the interval $[-1,4]$. Compute $L(f,P_5)$ and $U(f,P_5)$ where P_5 is the partition of $[-1,4]$ into 5-subintervals of equal length. What can you say about the values of $L(f,P_4)$ and $U(f,P_6)$ when you compare with $L(f,P_5)$ and $U(f,P_5)$?

Solution:





5-subint. of equal length.

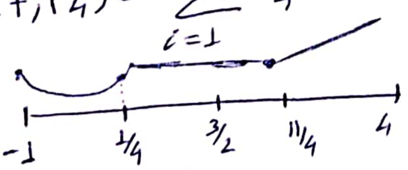
For $L(f,P_5)$ and $U(f,P_5)$, divide $[-1,5]$ into 5-subint. of equal length.

$$L(f,P_5)=\sum_{i=1}^{5} 1\cdot f(l_i) = 1.f(0)+1.f(0)+1.f(1)+1.f(2)+1.f(3)$$
$$= 2+2+3+3+4 = 14$$

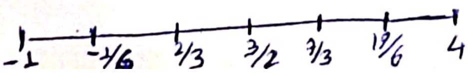
$$U(f,P_5)=\sum_{i=1}^{5} 1.f(u_i) = 1.f(-1)+1.f(1)+1.f(1)+1.f(3)+1.f(4)$$
$$= 3+3+3+4+6 = 19$$

For the partition P_4, divide $[-1,4]$ into 4-subint. of equal length.
Similarly, for P_6, " " 6 " .

$$L(f,P_4)=\sum_{i=1}^{4} \frac{5}{4}\cdot f(l_i) = \frac{5}{4}\cdot\left(f(0)+f(1/4)+f(3/2)+f(11/4)\right)$$
$$=\frac{5}{4}\cdot\left(2+\frac{33}{16}+3+\frac{7}{2}\right)=\frac{5}{4}\cdot\frac{169}{16}\cong 13.2$$



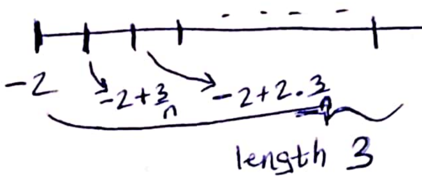
$$U(f,P_6)=\sum_{i=1}^{6}\frac{5}{6}\cdot f(u_i)=\frac{5}{6}\cdot\left(f(-1)+f(2/3)+f(3/2)+f(2)+f(19/6)+f(4)\right)=\frac{5}{6}\cdot\frac{196}{9}\cong 18.1$$
$$=\frac{5}{6}\cdot\left(3+\frac{22}{9}+3+3+\frac{13}{3}+6\right)$$



③ Write a Riemann sum with equal length subintervals to compute the integral

$$\int_{-2}^{1}\left(x^2+x\cdot\sin\left(\frac{1}{x+3}\right)\right)dx$$

Solution: We have an interval of length 3 and $f(x)=x^2+x\sin\left(\frac{1}{x+3}\right)$



Let us divide this interval n equal subintervals and take as c_i as the right endpoint of the subintervals, i.e., $c_i=-2+\frac{i}{n}\cdot 3$

where $P_n=\left\{-2+i\cdot\frac{3}{n}\right\}_{i=0}^{n}$.

$$R(f,P_n)=\sum_{i=1}^{n}f(c_i)\cdot\Delta x_i=\sum_{i=1}^{n}f\left(-2+\frac{3}{n}\cdot i\right)\cdot\frac{3}{n}$$

$$=\sum_{i=1}^{n}\left[\left(-2+\frac{3}{n}\cdot i\right)^2+\left(-2+\frac{3}{n}\cdot i\right)\cdot\sin\left(\frac{1}{\frac{3}{n}\cdot i+1}\right)\right]\cdot\frac{3}{n},$$

$$\lim_{n\to\infty}R(f,P_n)=\int_{-2}^{1}\left(x^2+x\cdot\sin\left(\frac{1}{x+3}\right)\right)dx$$

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④ Express the following limits as definite integrals.

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\arctan\left(3+\frac{2i}{n}\right)$ (b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\sin\left(\frac{2i-1}{2n}\right)$

c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n+2i}$. Compute this limit.

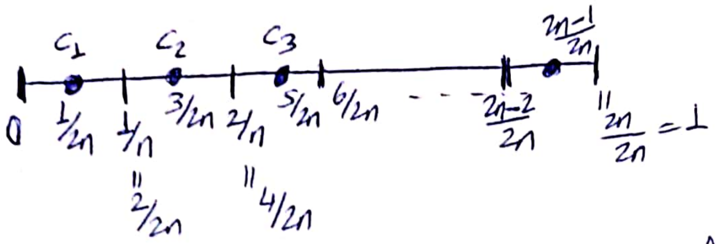
Solution: a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\arctan\left(3+2\cdot\frac{i}{n}\right)=\int_{0}^{1}\arctan(3+2x)$

$\Delta x_i=\frac{1}{n}$,

, $c_i=i/n$, where $P_n=\left\{\frac{i}{n}\right\}_{i=0}^{n}$

b) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \sin\left(\frac{2i-1}{2n}\right)$

for $i = 1, 2, 3, \dots, n$, we have $\left\{\frac{1}{2n}, \frac{3}{2n}, \frac{5}{2n}, \dots, \frac{2n-1}{2n}\right\}$.



So, divide $[0,1]$ into n subint. of equal length, but consider

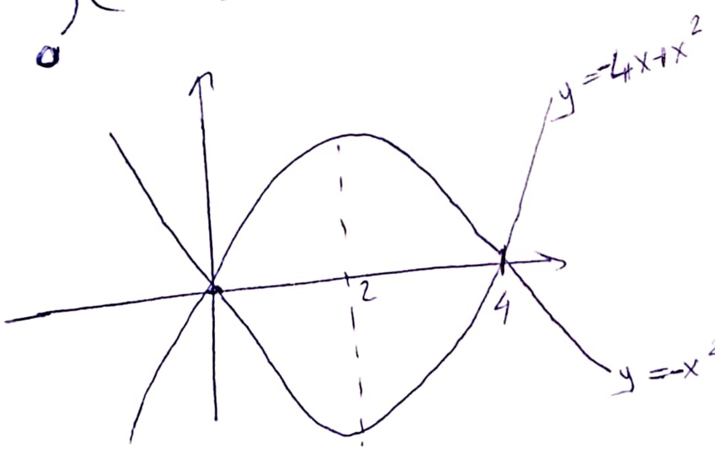
$c_i = \frac{x_{i-1} + x_i}{2}$ (as the middle pts of each subint.)

So, $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \sin\left(\frac{2i-1}{2n}\right) = \int_0^1 \sin x \, dx$

⑤ Find the values of a and b $(a<b)$ which maximize the integral

$$\int_a^b (4x - x^2)\,dx$$

Solution:



$y = 4x - x^2 = x(4-x)$

If $a \le 0, b \le 0$, then we get negative values from $\int_a^b (4x-x^2)\,dx$.

Similarly, if $a \geq 4, b \geq 4$, then " ".

In order to get a max. value both a and b should be between 0 and 4. In fact, $\int_0^4 (4x-x^2)\,dx$ is the max. value since it represents the max area.