MATH119 - Recit. - Week 7

① Evaluate the following limits

(a) $\lim_{x\to\infty} (\pi - 2\arctan x)\ln x$ (b) $\lim_{x\to\infty} \left[(x+2)e^{1/x} - x\right]$

(c) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ (d) $\lim_{x\to 1} \frac{x\ln|x| - x - 1}{(x+1)^2}$

(e) $\lim_{x\to\infty} \left(1 + \frac{2}{x} - \frac{3}{x^2}\right)^x$ (f) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Solution:

a) $\lim_{x\to\infty} (\pi - 2\arctan x)\ln x \overset{[0.\infty]}{=} \lim_{x\to\infty} \frac{\pi - 2\arctan x}{1/\ln x}$

$\underset{\text{L'hosp.}}{\overset{[\frac{0}{0}]}{=}} \lim_{x\to\infty} \frac{\frac{-2}{1+x^2}}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{2x.(\ln x)^2}{1+x^2}$

$\underset{\text{L'hosp.}}{\overset{[\frac{\infty}{\infty}]}{=}} \lim_{x\to\infty} \frac{2(\ln x)^2 + 2x.2\ln x.\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{2.(\ln x).(\ln x + 2)}{2x}$

$\underset{\text{L'h.}}{\overset{[\frac{\infty}{\infty}]}{=}} \lim_{x\to\infty} \frac{2.\frac{1}{x}(\ln x + 2) + \ln x.\frac{1}{x}.2}{2} = \lim_{x\to\infty} \left(2\frac{\ln x}{x} + \frac{2}{x}\right)$,

$\lim_{x\to\infty} \frac{\ln x}{x} \underset{\text{L'h.}}{\overset{[\frac{\infty}{\infty}]}{=}} \lim_{x\to\infty} 1/x = 0$ (Exists) and $\lim_{x\to\infty} \frac{2}{x} = 0$ (Exist)

So, $\lim_{x\to\infty} \left(2\frac{\ln x}{x} + \frac{2}{x}\right) = 0.$

b) $\lim_{x\to\infty}\left[(x+2).e^{1/x}-x\right] \underset{\boxed{\frac{1}{x}=t}}{=} \lim_{t\to 0}\left(\frac{1}{t}+2\right)e^{t}-\frac{1}{t}$

$= \lim_{t\to 0}\frac{(1+2t)e^{t}-1}{t} \underset{L'h.}{\overset{[\frac{0}{0}]}{=}} \lim_{t\to 0} 2e^{t}+(1+2t)e^{t} = 2+1=3$

(d) $\lim_{x\to -1}\frac{x\ln|x|-x-1}{(x+1)^2} \underset{\boxed{x+1=t}}{\overset{[\frac{0}{0}]}{=}} \lim_{t\to 0}\frac{(t-1)\ln|t-1|-t}{t^2}$

$\underset{L'h.}{\overset{[\frac{0}{0}]}{=}} \lim_{t\to 0}\frac{\ln|t-1|+\cancel{(t-1)}.\frac{1}{\cancel{t-1}}-1}{2t} = \lim_{t\to 0}\frac{\ln|t-1|-1}{2t}$

$\underset{L'h.}{\overset{[\frac{0}{0}]}{=}} \lim_{t\to 0}\left(\frac{1}{t-1}\right)\cdot\frac{1}{2} = -1/2$

(e) $\lim_{x\to\infty}\left(1+\frac{2}{x}-\frac{3}{x^2}\right)^{x} \rightarrow [1^{\infty}]$.

Let $y=\left(1+\frac{2}{x}-\frac{3}{x^2}\right)^{x}$. Then, $\ln y = x.\ln\left(1+\frac{2}{x}-\frac{3}{x^2}\right)$.

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x.\ln\left(1+\frac{2}{x}-\frac{3}{x^2}\right) \overset{[0.\infty]}{=} \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}-\frac{3}{x^2}\right)}{1/x}$

$\underset{L'h.}{\overset{[\frac{0}{0}]}{=}} \lim_{x\to\infty}\left(\frac{\left(-\frac{2}{x^2}+\frac{6}{x^3}\right)}{\frac{x^2+2x-3}{x^2}}\right)\cdot(-x^2) = \lim_{x\to\infty}\frac{6-2x}{x^3}\cdot\frac{x^2}{x^2+2x-3}\cdot(-x^2)$

$= -\lim_{x\to\infty}\frac{6x-2x^2}{x^2+2x-3} = 2$, And since the exp. func. e^{x} is cont.,

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^{2}$

(2) Find the absolute max and min. values of $f(x)=\frac{x^2+4}{x}$ on the following intervals if exist

(a) $[-1,3]$ (b) $[1,4]$ (c) $(-\infty,-2]$

(d) $(-2,\infty)$ (e) $(-\infty,\infty)$

Solution: Recall: (Extreme Value Thm (E.V.T.))

If $f(x)$ is cont. on a closed finite interval or a union of finitely many such intervals, then f must have an abs. max. value and an abs. min. value

Another thm also says that if the func. f is defined on an interval I and has a local max. (or min) value, then it must be a critical point or singular point or an endpoint.

a) $f(x)=\frac{x^2+4}{x}=x+\frac{4}{x}$ is defined on $[-1,0)\cup(0,3]$.

$f(x)$ is also cont. on $\mathbb{R}-\{0\}$.

$f'(x)=1-\frac{4}{x^2}=\frac{x^2-4}{x^2}=0 \Rightarrow x=\pm 2 \rightarrow$ cr. pts.

$x=0 \rightarrow$ sing. point.

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\frac{x^2+4}{x}=\lim\limits_{x\to 0^-} x+\frac{4}{x}=-\infty$ and $\lim\limits_{x\to 0^+}\frac{x^2+4}{x}=\infty$.

So, f does not have any abs. max or min on $[-1,3]$.

b) $[1,4]$ is a closed finite interval and $f(x)$ is cont. on it.

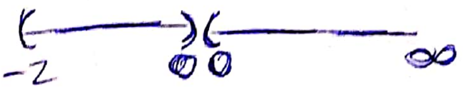
So, $f(x)$ must have abs. extr. To find them, we need to compare cr. pts, sing. pts and endpoints.

$\underbrace{f(2)=4}_{\text{abs. min}}$, $\underbrace{f(1)=5}_{\text{abs. max}}$, $f(3)=13/3$

d) $(-2,\infty) \longrightarrow (-2,0)\,,\,(0,\infty)$ since $f(x)$ is not defined at $x=0$.

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2+4}{x} = -4$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{x^2+4}{x} = \infty .$$



$$\lim_{x\to 0^+} f(x) = \infty \;, \quad \lim_{x\to 0^-} f(x) = -\infty$$

		-2		0		2	
f'	$+$	0	$-$	$\Vert$	$-$	0	$+$
f	$\nearrow$		$\searrow$	$\Vert$	$\searrow$		$\nearrow$

$f(x)$ does not have an abs. min on $(-2,0)$.

Since $f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2}$ is decr. on $(-2,0)$, also there does not exist an abs. max.

Since $\lim_{x\to\infty} f(x) = \infty$, there is no abs. max on $(0,\infty)$.

And since $\lim_{x\to 0^+} f(x) = \infty$ and $f(2)=4$, $x=2$ is abs. min.

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③ Determine whether the given functions has any local or abs. extr. values, and those values if possible.

(a) $f(x) = |x^2-x-6|$ on $[-3,3]$

(b) $f(x) = |x^2-x-6|$

(c) $f(x) = \dfrac{x}{\sqrt{x^4+1}}$

(d) $f(x) = x - 2\arctan x$

Solution: a) Since $f(x)$ is cont. on a closed finite interval, there exist abs. extr.

On $[-3,-2]$: $f(x) = x^2-x-6 \Rightarrow f'(x) = 2x-1 = 0 \Rightarrow x = 1/2$ ×

and On $[-2,3]$: $f(x) = -x^2+x+6 \Rightarrow f'(x) = -2x+1 = 0 \Rightarrow x = 1/2$ ✓

	-3		-2		$1/2$		3
f'		$-$		$+$		$-$	
f		$\searrow$		$\nearrow$		$\searrow$	

$f(-3)=6$, $\underbrace{f(-2)=0}_{\text{local (abs) min}}$ (sing. pt.), $\underbrace{f(1/2)=25/6}_{\text{local (abs) max}}$, $\underbrace{f(3)=0}_{\text{abs min}}$

b) $\lim\limits_{x\to-\infty} f(x) = \infty = \lim\limits_{x\to\infty} f(x)$, so there is no abs. max.

Since extr. values occur when the point is critical, singular or ~~endpoint~~, $x=-2$ and $x=3$ are abs. min.

c) $f(x) = \dfrac{x}{\sqrt{x^4+1}}$ is defined and cont. everywhere.

$$f'(x) = \frac{\sqrt{x^4+1} - x \cdot \dfrac{4x^3}{2\sqrt{x^4+1}}}{x^4+1} = \frac{2x^4+2-4x^4}{2(x^4+1)^{3/2}}$$

$$= \frac{1-x^4}{(x^4+1)^{3/2}} = 0 \Rightarrow x = \pm 1 \rightarrow \text{cr. pts.}$$

No singular pt.

		-1		1	
f'	$-$		$+$		$-$
f	↘		↗		↘

$x=-1 \rightarrow$ local min.

$x=1 \rightarrow$ local max.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^4+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^4(1+\frac{1}{x^4})}}$$

$$= \lim_{x\to\infty} \frac{x}{x^2\sqrt{1+1/x^4}} = 0 = \lim_{x\to-\infty} f(x)$$

Since there are values of f greater than 0, there must be abs. max, and since there are values of f less than 0, there must be abs. min. And since extr. values occur on the cr. points or sing. pts or endpoints, $f(-1) = -\frac{1}{\sqrt{2}}$, $f(1) = \frac{1}{\sqrt{2}}$

$f(-1) \rightarrow$ abs. min

$f(1) \rightarrow$ abs. max.

④ Find and classify all local extr. values of $f(x)$. Determine whether any of these extr. values are absolute. Find the intervals on which $f(x)$ is incr., decr., concave up and down.
Locate any inflection points if exist.

(a) $f(x)=x\sqrt{4-x^2}$ (b) $f(x)=(x^2-x-2)e^x$

(c) $f(x)=\dfrac{x}{\ln x}$ (d) $f(x)=x+\sin x$

Solution: a) $4-x^2\geq 0 \Rightarrow x^2\leq 4 \Rightarrow -2\leq x\leq 2$. So,

$\mathrm{Dom}(f(x))=[-2,2]$.

$$f'(x)=\sqrt{4-x^2}+x.(-2x).\frac{1}{2\sqrt{4-x^2}}=\frac{8-2x^2-2x^2}{2\sqrt{4-x^2}}=$$

$$=\frac{4-2x^2}{\sqrt{4-x^2}}=0\Rightarrow x=\pm\sqrt{2}\rightarrow \text{cr. pts.}$$
$$x=\pm 2\rightarrow \text{sing. pts.}$$

$$f''(x)=\frac{(-4x).\sqrt{4-x^2}-(4-2x^2).(-2x).\frac{1}{2\sqrt{4-x^2}}}{4-x^2}=\frac{-8x(4-x^2)+8x-4x^3}{2.(4-x^2)^{3/2}}$$

$$=\frac{2x^3-12x}{(4-x^2)^{3/2}}=0\Rightarrow 2x(x^2-6)=0\Rightarrow x=0\in[-2,2]$$
$$x=\pm\sqrt{6}\ \times$$

	-2		$-\sqrt{2}$		0		$\sqrt{2}$		2
f'		$-$	0	$+$		$+$	0	$-$	
f''		$+$		$+$	0	$-$		$-$	
f		decr. conc. up		incr. conc. up		incr. conc. down		decr. conc. down	

$x=-\sqrt{2}\rightarrow$ local min $\quad x=0\rightarrow$ infl. pt.

$x=\sqrt{2}\rightarrow$ local max.