MATH119 - 2021 Fall - Recit. Week 3

1. Use the formal ($\varepsilon - \delta$) definition of the limit to prove

(a) $\lim_{x \to 3} (7x-6) = 15$ (b) $\lim_{x \to 0} (x^2 \sin(x^2-2x)+2) = 2$

(c) $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$

Solution: a) Let $\varepsilon > 0$. Need to find some $\delta > 0$ such that

$$|x-3| < \delta \Rightarrow |(7x-6)-15| < \varepsilon.$$

So, $|(7x-6)-15| = |7x-21| = 7|x-3| < 7 \cdot \delta = \varepsilon.$

Then, choose $\delta = \varepsilon/7$.

b) Let $\varepsilon > 0$. Find some $\delta > 0$ such that

$$|x-0| < \delta \Rightarrow |(x^2\sin(x^2-2x)+2)-2| < \varepsilon.$$

Let $|x| < \delta$.

$$|(x^2\sin(x^2-2x)+2)-2| = |x^2\sin(x^2-2x)| = \underbrace{|x|^2}_{<\delta^2} \cdot \underbrace{|\sin(x^2-2x)|}_{\leq 1} \leq \delta^2 < \varepsilon$$

So, by choosing $\delta < \sqrt{\varepsilon}$, specifically let $\delta = \frac{\sqrt{\varepsilon}}{2}$,

we have proven that $\lim_{x \to 0} (x^2\sin(x^2-2x)+2) = 2$.

c) Let $\varepsilon > 0$. Find some $\delta > 0$ such that

$$|x-0| < \delta \Rightarrow |x\sin(\tfrac{1}{x})| < \varepsilon.$$

$|x\sin(\frac{1}{x})| = \underbrace{|x|}_{<\delta} \cdot \underbrace{|\sin(\frac{1}{x})|}_{\leq 1} \leq \delta < \varepsilon$. So, choose $\delta = \varepsilon/2$.

for all $x \neq 0$.

② Compute $f'(x)$ using the limit definition of the derivative for the following functions

(a) $f(x) = x^2+5x$ (b) $f(x)=x-\sqrt{x}$

Solution: a) Recall that the derivative of $f(x)$ at the point $x=x_0$ is

$$f'(x_0) = \left.\frac{d}{dx}f(x)\right|_{x=x_0} = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \underset{\boxed{x-x_0=h}}{=} \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

So, $f'(x_0) = \lim\limits_{x\to x_0}\dfrac{x^2+5x-(x_0^2+5x_0)}{x-x_0} = \lim\limits_{x\to x_0}\dfrac{x^2-x_0^2+5x-5x_0}{x-x_0}$

$= \lim\limits_{x\to x_0}\dfrac{\cancel{(x-x_0)}\cdot(x+x_0+5)}{\cancel{x-x_0}} = 2x_0+5$, So $f'(x)=2x+5$

b) $f'(x_0) = \lim\limits_{x\to x_0}\dfrac{x-\sqrt{x}-(x_0-\sqrt{x_0})}{x-x_0} = \lim\limits_{x\to x_0}\dfrac{x-x_0+\sqrt{x_0}-\sqrt{x}}{x-x_0}$

$= \lim\limits_{x\to x_0}\dfrac{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})-(\sqrt{x}-\sqrt{x_0})}{x-x_0}$

$= \lim\limits_{x\to x_0}\dfrac{\cancel{(\sqrt{x}-\sqrt{x_0})}(\sqrt{x}+\sqrt{x_0}-1)}{\cancel{(\sqrt{x}-\sqrt{x_0})}(\sqrt{x}+\sqrt{x_0})} = \dfrac{2\sqrt{x_0}-1}{2\sqrt{x_0}} = 1-\dfrac{1}{2\sqrt{x_0}}$,

So, $f'(x) = 1-\dfrac{1}{2\sqrt{x}}$

③ In the following, each limit represents a derivative $f'(a)$. Find a func. $f(x)$ and a point a.

(a) $\lim\limits_{h\to 0} \frac{(5+h)^3-125}{h}$ (b) $\lim\limits_{h\to 0} \frac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h}$

(c) $\lim\limits_{x\to 1/4} \frac{x^{-1}-4}{x-1/4}$

Solution: a) Recall that $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

So, let $f(x) = x^3$ and $x_0 = 5$. Then,

$$f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} = \lim_{h\to 0} \frac{(5+h)^3-125}{h}$$

b) Let $f(x) = \sin x$ and $x_0 = \pi/6$. Then,

$$f'(\pi/6) = \lim_{h\to 0} \frac{f(\pi/6+h)-f(\pi/6)}{h} = \lim_{h\to 0} \frac{\sin(\pi/6+h)-1/2}{h}$$

c) Let $f(x) = x^{-1}$ and $x_0 = 1/4$. Then,

$$f'(1/4) = \lim_{x\to 1/4} \frac{f(x)-f(1/4)}{x-1/4} = \lim_{x\to 1/4} \frac{x^{-1}-4}{x-1/4}$$

④ Let $f(x)=|x|$ and $g(x)=x.|x|$. Show that $f'(0)$ does not exist; however, $g'(0)$ exists and equals to 0. Use the definition of derivative.

Solution: $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|}{h}$ does not exist,

since $\lim_{h\to 0^+}\frac{|h|}{h} \underset{h>0}{=} \lim_{h\to 0^+} h/h = 1 \neq -1 = \lim_{h\to 0^-}\frac{|h|}{h} \underset{h<0}{=} \lim_{h\to 0^-}\frac{-h}{h}$

So, $f'(0)$ D.N.E.

$$g'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{h.|h|-0}{h}=\lim_{h\to 0}|h|=0.$$

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⑤ Find $f'(0)$ if

$$f(x)=\begin{cases}\frac{x^3\sin(1/x)}{\sin x} & , \text{if } x\neq 0\\ 0 & , \text{if } x=0.\end{cases}$$

Solution: $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^3\sin(1/h)}{\sinh}\cdot\frac{1}{h}$

$$=\lim_{h\to 0}\frac{h.\sin(1/h).h}{\sinh}$$

Recall that $\lim_{h\to 0}\frac{\sinh}{h}=1$, so $\lim_{h\to 0}\frac{h}{\sinh}=1$. (Exists)

By squeeze thm,

$0\leq|\sin(1/h).h|\leq|h|$, for all $h\neq 0$. By squeeze thm, $\lim_{h\to 0}\sin(1/h).h=0$. (Exists)

$0=\lim_{h\to 0}0=\lim_{h\to 0}|h|=\lim_{h\to 0}|\sin(1/h).h|$. Thus $\lim_{h\to 0}\sin(1/h).h=0$.

So, $\lim_{h\to 0}\frac{h}{\sinh}\cdot\sin(1/h).h=\lim_{h\to 0}\frac{h}{\sinh}\cdot\lim_{h\to 0}\sin(1/h).h=0$.

⑥ Find a and b so that the func.

$$f(x)=\begin{cases} \tan x & , \text{ if } x<\pi/6 \\ a+bx & , \text{ if } x\geq \pi/6 \end{cases}$$

is differentiable at $x=\pi/6$.

Solution: For differentiability we first need the continuity at the same point. (Recall that differentiability implies continuity but we may not have the other way around.)

$$\lim_{x\to\pi/6^-} f(x) = \lim_{x\to\pi/6^-} \tan x = \frac{\sqrt{3}}{3} = \lim_{x\to\pi/6^+} f(x) = \lim_{x\to\pi/6} a+bx = a+\frac{\pi}{6}b$$

$$= f(\pi/6) \Longrightarrow \boxed{a+\frac{\pi}{6}b=\frac{\sqrt{3}}{3}} \quad (*)$$

Now check diff. at $x=\pi/6$

$$f'(\pi/6) = \lim_{x\to\pi/6} \frac{f(x)-f(\pi/6)}{x-\pi/6} = \lim_{x\to\pi/6} \frac{f(x)-(a+\frac{\pi}{6}b)}{x-\pi/6},$$

$$\lim_{x\to\pi/6^+} \frac{f(x)-(a+\frac{\pi}{6}b)}{x-\pi/6} = \lim_{x\to\pi/6^+} \frac{a+bx-a-\frac{\pi}{6}b}{x-\pi/6} = b \quad \text{and}$$

$$\lim_{x\to\pi/6^-} \frac{f(x)-(a+\frac{\pi}{6}b)}{x-\pi/6} = \lim_{x\to\pi/6} \frac{\tan x - a - \frac{\pi}{6}b}{x-\pi/6} \overset{(*)}{=} \lim_{x\to\pi/6^-} \frac{\tan x - \sqrt{3}/3}{x-\pi/6}$$

$$= \lim_{h\to0^-} \frac{\tan(\frac{\pi}{6}+h)-\tan(\pi/6)}{h} = \lim_{h\to0^-} \frac{\tan h\cdot(1+\tan(\frac{\pi}{6}+h)\cdot\tan(\frac{\pi}{6}))}{h}$$

$$= 1\cdot\left(1+\left(\frac{\sqrt{3}}{3}\right)^2\right) = \frac{4}{3} = b$$

$$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta} \quad \text{and} \quad x-\frac{\pi}{6}=h}$$

So, $a+\frac{\pi}{6}\cdot\frac{4}{3}=\frac{\sqrt{3}}{3} \Rightarrow a=\frac{\sqrt{3}}{3}-\frac{2\pi}{9}$

⑦ For the following functions, determine the points c (if any) such that $f'(c)$ does not exist.

(a) $f(x) = x^{3/2}$ (b) $f(x) = |x-2|^2$ (c) $f(x) = |x^2-1|$

Solution: a) $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

$$= \lim_{h\to 0} \frac{((c+h)^{3/2} - c^{3/2}).((c+h)^{3/2}+c^{3/2})}{h.((c+h)^{3/2}+c^{3/2})} = \lim_{h\to 0} \frac{(c+h)^3 - c^3}{h.((c+h)^{3/2}+c^{3/2})}$$

$$= \lim_{h\to 0} \frac{3c^2h+3ch^2+h^3}{h((c+h)^{3/2}+c^{3/2})} = \lim_{h\to 0} \frac{\cancel{h}(3c^2+3ch+h^2)}{\cancel{h}((c+h)^{3/2}+c^{3/2})}$$

$= \frac{3c^2}{2c^{3/2}} = \frac{3}{2}c^{1/2}$, so for all $c \in Dom(f(x))$, $f'(c)$ exists.

(b) $f'(c) = \lim_{x\to c} \frac{|x-2|^2 - |c-2|^2}{x-c} = \lim_{x\to c} \frac{(|x-2|-|c-2|).(|x-2|+|c-2|)}{x-c}$

when $c>2$;

$$\lim_{x\to c} \frac{(|x-2|-(c-2))(|x-2|+c-2)}{x-c} = \lim_{x\to c} \frac{(\cancel{x-2-c+2})(x-2+c-2)}{\cancel{x-c}} = 2c-4$$

when $c<2$;

$$\lim_{x\to c} \frac{(|x-2|+c-2)(|x-2|-c+2)}{x-c} = \lim_{x\to c} \frac{(\cancel{-x+2+c-2}).(-x+2-c+2)}{\cancel{x-c}} = -4+2c$$

when $c=2$;

$$\lim_{x\to 2} \frac{|x-2|.|x-2|}{x-2}$$

$\lim_{x\to 2^+} \frac{(x-2)^2}{x-2} = 0$

$\lim_{x\to 2^-} \frac{(-(x-2))^2}{x-2} = 0$

} so, $f'(c)$ exists for all c.

8 Prove each of the following.

(a) The derivative of an even func. is an odd func.

(b) " " odd " " even "

Solution: a) Let $f(x)$ be an even func. So, $f(-x)=f(x)$ for all x.

Let $g(x_0)=f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

Then, $g(-x_0)=\lim_{x\to -x_0}\frac{f(x)-f(-x_0)}{x-(-x_0)}=\lim_{x\to -x_0}\frac{f(x)-f(x_0)}{x+x_0}$

$\underset{\boxed{x=-y}}{=}\lim_{y\to x_0}\frac{f(-y)-f(x_0)}{x_0-y}=\lim_{y\to x_0}\frac{f(y)-f(x_0)}{-(y-x_0)}$

$\underset{\boxed{y=x}}{=}-\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=-g(x_0)$. So, $g(x)$ is an odd func.

b) Similarly.