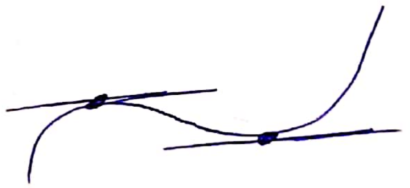
MATH 119 – Recit. Problems – Week 4

① Find the points on the graph of $f(x) = 12x - x^3$ where the tangent line is horizontal.

Solution:



If the tangent line of $f(x)$ at a point (x_0, y_0) is horizontal, then the slope of this line is zero. So,

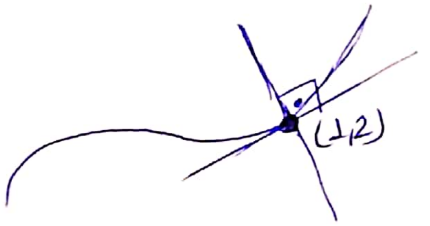
$$f'(x_0) = 12 - 3x_0^2 = 0 \Rightarrow x_0^2 = 4 \Rightarrow x_0 = \pm 2$$

$$\Rightarrow y_0 = f(x_0) = 12.(-2) - (-2)^3 \quad \text{and} \quad 12.2 - 2^3 = 16$$
$$= -16$$

So, the points we need are $(-2, -16)$ and $(2, 16)$

② Find the equations of the tangent and normal line at the point $(1, 2)$ to the graph of $f(x) = -4x^2 + 6x$

Solution:



$f'(x) = -8x + 6 \Rightarrow f'(1) = -2$. So, the slope of the tangent line, $m_1 = -2$. Since the normal line is perpendicular to the tang. line, the slope of the normal line $m_2 = \frac{-1}{-2} = \frac{1}{2}$ $(m_1 \cdot m_2 = -1)$

An eqn. for the tang. line is

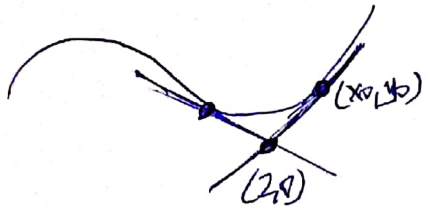
$$y - 2 = (-2).(x - 1) \Rightarrow y = -2x + 4$$

An eqn. for the normal line is

$$y - 2 = \frac{1}{2}.(x - 1) \Rightarrow y = \frac{x}{2} + \frac{3}{2}$$

③ Find two straight lines that are tangent to the curve $y=\frac{x^2}{x-1}$ and pass through the point $(2,0)$.

Solution:



$(2,0)$ is not on this curve since $0 \neq \frac{2^2}{2-1}$.

Let (x_0,y_0) be on this curve. Then the line passing thr. (x_0,y_0) and $(2,0)$ has slope $m=\frac{y_0-0}{x_0-2}$ which is $f'(x_0)$.

So, $f'(x)=\frac{2x(x-1)-x^2.1}{(x-1)^2}=\frac{2x^2-2x-x^2}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}$

$\Rightarrow f'(x_0)=\boxed{\frac{x_0^2-2x_0}{(x_0-1)^2}}=\frac{y_0}{x_0-2}=\boxed{\frac{x_0^2}{x_0-1}\cdot\frac{1}{x_0-2}}$

$\Rightarrow x_0(x_0-2)^2.(x_0-1)=x_0^2(x_0-1)^2 \Rightarrow x_0.(x_0-1).[(x_0-2)^2-x_0(x_0-1)]=0$

$\Rightarrow x_0(x_0-1).(-3x_0+4)=0 \Rightarrow$ $x_0=0$, $x_0=1$ → Not in the domain. $x_0=4/3$

$x_0=0 \Rightarrow y_0=0$

$x_0=4/3 \Rightarrow y_0=\frac{16}{9}.3=\frac{16}{3}$ $\Rightarrow$ The points are $(0,0)$ and $(\frac{4}{3},\frac{16}{3})$

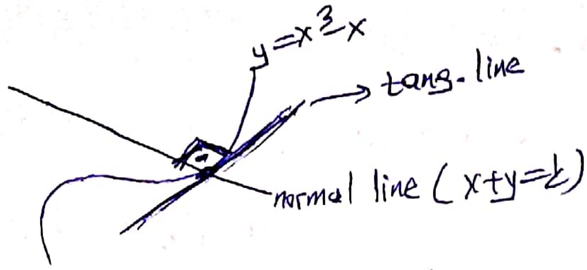
For the first line, passing thr. $(0,0)$, the slope is 0. So, an eqn is $\underline{y=0}$

For the second line, passing thr. $(\frac{4}{3},\frac{16}{3})$, the slope is $\frac{16}{3}.(\frac{-3}{2})=-8$.

So, an eqn is $y-\frac{16}{3}=(-8).(x-\frac{4}{3})$ or $y-0=(-8).(x-2)$.

(4) For what values of k, is the line $x+y=k$ normal to the curve $y=x^3-x$?

Solution:



Let (x_0, y_0) be on the curve.

The slope of the tang. line to this curve at (x_0, y_0) is $f'(x_0)$. So, $f'(x)=3x^2-1 \Rightarrow f'(x_0)=3x_0^2-1$.

Since the tang. line is normal to the line $x+y=k$, which has slope -1 at this point, $(3x_0^2-1)\cdot(-1)=-1 \Rightarrow 3x_0^2-1=1$

$\Rightarrow x_0^2=\frac{2}{3} \Rightarrow x_0=\pm\sqrt{\frac{2}{3}}$.

$x_0=-\sqrt{\frac{2}{3}} \Rightarrow y_0=-\frac{2}{3}\cdot\sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{3}=\frac{\sqrt{2}}{3\sqrt{3}}$

$x_0=\frac{\sqrt{2}}{\sqrt{3}} \Rightarrow y_0=\frac{2}{3}\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}=-\frac{\sqrt{2}}{3\sqrt{3}}$

$x_0+y_0=k \Rightarrow k=-\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{3\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{2}{3}\right)$ and

$x_0+y_0=k \Rightarrow k=\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt{2}}{3\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{2}{3}$

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(5) Suppose that $f(2)=3$, $f'(2)=4$, $g(2)=5$ and $g'(2)=6$. Find

(a) $(fg)'(2)$ (b) $(f^2g)'(2)$ (c) $(f/g^2)'(2)$

Solution: b) $2f(2)\cdot f'(2)\cdot g(2)+f^2(2)\cdot g'(2)$ (product rule with chain rule)

c) $\dfrac{f'(2)g^2(2)-f(2)\,2g(2)g'(2)}{g^4(2)}$

(6) Find $F'(x_0)$ by using the given information.

(a) $F(x) = f(2f(4f(x)))$, $x_0 = 0$, $f(0) = 0$, $f'(0) = 2$

(b) $F(x) = f(xf(x))$, $x_0 = 1$, $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$.

Solution: a) $F'(x_0) = f'(2f(4f(x_0))) \cdot 2 \cdot f'(4f(x_0)) \cdot 4f'(x_0)$

$\Rightarrow F'(x_0) = F'(0) = f'(2f(4 \cdot 0)) . 2f'(4 \cdot 0) . 4f'(0)$

$= f'(0) . 2 . f'(0) . 4f'(0) = 2 . 4 . 8 = 64$

(7) Find the derivative $\frac{dy}{dx}$ for each of the following functions:

(a) $y = \sqrt{x + \sqrt{1+x}}$ (b) $y = \left(\frac{2x+3}{4x+5}\right)^6$ (c) $y = \frac{x^2 + 3\sqrt[3]{x}}{x^3 + \sqrt{x}}$

(d) $y = \sqrt{x}(1 - x + x^2 - x^3)(1+x)$ (e) $y = x^3 \cos(x^2 + x)$

(f) $y = \sin(\sin(\sin(x)))$ (g) $y = \sec(x^3)^2 \sin(x^3)$

(h) $y = \tan^2(\sec(x^2))$

Solution: a) $y' = \frac{dy}{dx} = \left(\left(x + (1+x)^{1/2}\right)^{1/2}\right)' = \frac{1}{2\sqrt{x + \sqrt{1+x}}} \cdot \left(1 + \frac{1}{2\sqrt{1+x}}\right)$

c) $y' = \frac{\left(2x + \frac{1}{3} \cdot x^{-2/3}\right)(x^3 + \sqrt{x}) - (x^2 + 3\sqrt{x}) . (3x^2 + \frac{1}{2\sqrt{x}})}{(x^3 + \sqrt{x})^2}$

f) $y' = \cos(\sin(\sin x)) . \cos(\sin x) . \cos x$

h) $y' = 2 . \tan(\sec(x^2)) . \sec^2(\sec(x^2)) . \sec(x^2) . \tan(x^2) . 2x$

(8) Find all points on the graph of the function $f(x)=\sin(2x)-2\sin x$ at which the tangent line is horizontal.

Solution: $f'(x)=\cos(2x).2-2\cos x=0$

$\Rightarrow (2\cos^2 x-1).2-2\cos x=0$

$\Rightarrow 4\cos^2 x-2\cos x-2=0$, let $\cos x=t$.

$\Rightarrow 4t^2-2t-2=0 \Rightarrow (4t+2)(t-1)=0$

4t +2

t -1

$\Rightarrow \cos x=1$ or $\cos x=-\frac{1}{2}$

$\Downarrow$ $\Downarrow$

$x=2n.\pi$, $x=(\pi-\frac{\pi}{6})+2m\pi$

$x=(\pi+\frac{\pi}{6})+2k\pi$

(9) Show that the curves $xy=\sqrt{2}$ and $x^2=2y$ intersect at $P(\sqrt{2},1)$ and their tangent lines at P are perpendicular at each other.

(10) Calculate enough derivatives of the given func. to enable you to guess the general formula for $f^{(n)}(x)$.

(a) $f(x)=\frac{1}{3-x}$ (b) $f(x)=\sqrt{x}$ (c) $f(x)=x\cos x$

Solution: c) $f(x)=x\cos x$

$f'(x)=\cos x-x\sin x$

$f''(x)=-\sin x-\sin x-x\cos x=-2\sin x-x\cos x$

$f'''(x)=-2\cos x-\cos x+x\sin x=-3\cos x+x\sin x$

$f^{IV}(x)=3\sin x+\sin x+x\cos x=4\sin x+x\cos x$

$f^{(5)}(x)=4\cos x+\cos x-x\sin x=5\cos x-x\sin x$

$f^{(6)}(x)=-5\sin x-\sin x-x\cos x=-6\sin x-x\cos x$

$\Rightarrow f^{(2n)}(x)=((2n).\sin x+x\cos x).(-1)^n$, $f^{(2n+1)}(x)=((2n+1)\cos x-x\sin x).(-1)^n$