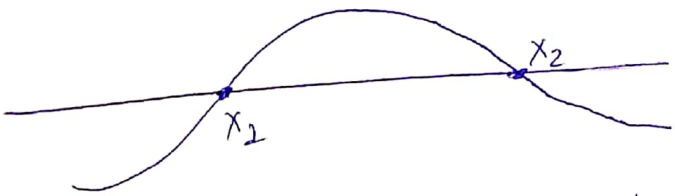
MATH119 – Recit. – Week 5

① Let f be a diff. func. on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for any $x \in \mathbb{R}$. Prove that f has at most one real root.

Solution: If $f(x)$ does not have any root, then there is nothing to prove. So, assume that $f(x)$ has at least one real root. On the contrary, suppose that $f(x)$ has two distinct real roots, say x_1 and x_2, which means

$f(x_1) = 0$, $f(x_2) = 0$.



Since $f(x)$ is diff. on $\mathbb{R}$, it is also diff. on (x_1, x_2) and cont. on $[x_1, x_2]$. Also $f(x_1) = f(x_2)$ $(=0.)$. So, by Rolle's thm, there is some $c \in (x_1, x_2)$ such that

$f'(c) = 0$.

But this contradicts with the fact that $f'(x) \neq 0$ for any $x \in \mathbb{R}$.

So, there should be at most one real root.

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② Show that $x^3 + x^2 + 3x + 7 = 0$ has exactly one real root.

Solution: First, let us show that there is at least one real root:

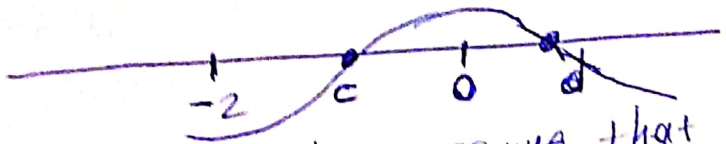
Let $f(x) = x^3 + x^2 + 3x + 7$.

$f(0) = 7 > 0$

$f(-2) = -8 + 4 - 6 + 7 = -3 < 0$

And since $f(x)$ is a polynomial, it is cont. on $[-2, 0]$. Thus, by IVT, there is some $c \in (-2, 0)$ such that $f'(c) = 0$.

So, we have at least one real root.



Without loss of generality, assume that there is some $d > c$ such that $f(d)=0$.

Since $f(x)$ is a poly., it is cont. on $[c,d]$ and diff. on (c,d). And also $f(c)=f(d)$ $(=0)$. So, by using Rolle's thm, there is some $e \in (c,d)$ such that

$$f'(e)=0.$$

$$\Rightarrow f'(x)=3x^2+2x+3 \Rightarrow f'(e)=3e^2+2e+3=0$$

But $\Delta = 4-4\cdot 3\cdot 3<0$, No root. So, there should not be any other root.

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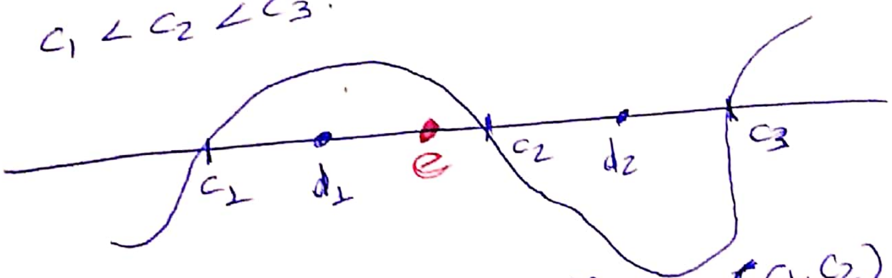
(3) Show that $6x^4-7x+1$ does not have more than two distinct real roots.

Solution: On the contrary, assume that $f(x)=6x^4-7x+1$ has 3 distinct roots, say $f(c_1)=f(c_2)=f(c_3)=0$ with $c_1<c_2<c_3$.



Since $f(x)$ is a poly, it is diff. on (c_1,c_2) an (c_2,c_3) and it is also cont. on $[c_1,c_2]$ and $[c_2,c_3]$. Also $f(c_1)=f(c_2)$, $f(c_2)=f(c_3)$. So, by Rolle's thm, there are some $d_1 \in (c_1,c_2)$ and $d_2 \in (c_2,c_3)$ s.t. $f'(d_1)=0$ and $f'(d_2)=0$.

Again, $f'(x) = 24x^3 - 7$ is cont. on $[d_1, d_2]$ and diff. on (d_1, d_2), also $f'(d_1) = f'(d_2)$. Thus, there is some $e \in (d_1, d_2)$ such that (by Rolle's thm).

$$(f')'(e) = 0.$$

$$\Rightarrow 72e^2 = 0 \Rightarrow e = 0.$$

So, except $e=0$, $f'' > 0$, thus f' is increasing on $\mathbb{R} - \{0\}$. But $f'(d_1) = 0$ and $f'(d_2) = 0$. Contradiction.
Therefore $f(x) = 6x^4 - 7x + 1$ does not have more than two distinct real roots.

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4) By using the MVT, show that $\tan x > x$ for $0 < x < \pi/2$.

<u>Solution:</u> Let $f(x) = \tan x$ and $x_0 \in (0, \pi/2)$.

$f(x) = \tan x$ is cont. on $[0, x_0]$ and diff. on $(0, x_0)$.

So, by the MVT, there is some $c \in (0, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0}.$$

$$\Rightarrow f'(c) = 1 + \tan^2 c = \underbrace{\frac{1}{\cos^2(c)}}_{>1} = \frac{\tan(x_0)}{x_0}$$

Since $0 < \cos^2(c) < 1$ for any $c \in (0, \frac{\pi}{2})$, $\frac{1}{\cos^2(c)} > 1$.

So, $\frac{\tan(x_0)}{x_0} > 1. \Rightarrow \tan(x_0) > x_0$ for any $x_0 \in (0, \frac{\pi}{2})$.

5. Suppose that $f(x)$ is cont. on $[-7,0]$, diff. on $(-7,0)$ and suppose that $f(-7)=-3$ and $f'(x)\leq 2$ for any $x\in(-7,0)$. What is the largest value for $f(0)$?

Solution: Since $f(x)$ is cont. on $[-7,0]$ and diff. on $(-7,0)$, by the MVT, there is some $c\in(-7,0)$ such that $f'(c)=\dfrac{f(0)-f(-7)}{0-(-7)}=\dfrac{f(0)+3}{7}$

Since for any $x\in(-7,0)$ $f'(x)\leq 2$,

$$f'(c)=\frac{f(0)+3}{7}\leq 2 \Rightarrow f(0)+3\leq 14$$
$$f(0)\leq 11.$$

So, the largest possible value for $f(0)$ is 11.

6. Find the intervals of increase and decrease of the following func.

(a) $f(x)=x^3+3x^2-x+1$ (b) $f(x)=x+2\cos x$

Solution: b) $f'(x)=1-2\sin x=0 \Rightarrow \sin x=1/2$

$\Rightarrow x=\frac{\pi}{6}+2n\pi$, $x=\frac{5\pi}{6}+2m\pi$, $n,m\in\mathbb{Z}$

	$\ldots$ $\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\ldots$
f'	$-$	$+$	$-$	$+$	
f	↘	↗	↘	↗	

So, $f(x)$ is incr. on $\ldots\cup\left(\frac{5\pi}{6},\frac{13\pi}{6}\right)\cup\left(\frac{17\pi}{6},\frac{25\pi}{6}\right)\cup\ldots=\bigcup_{n\in\mathbb{Z}}\left(\frac{5\pi}{6}+2n\pi,\frac{13\pi}{6}+2n\pi\right)$

decr. on $\ldots\cup\left(\frac{\pi}{6},\frac{5\pi}{6}\right)\cup\left(\frac{13\pi}{6},\frac{17\pi}{6}\right)\cup\ldots$

7 Find the equations of the tangent line and normal line drawn to the graph of $y=y(x)$ given implicitly by $x+\tan\left(\frac{y}{x}\right)=2$ at the point $P=(1,\pi/4)$

Solution: In order to find the slopes of the tangent and normal lines, we need to find $\left.\frac{dy}{dx}\right|_{(1,\pi/4)}$.

$$\Rightarrow 1+\sec^2\left(\frac{y}{x}\right)\cdot\left.\left(\frac{\left(\frac{dy}{dx}\right)\cdot x-y}{x^2}\right)\right|_{(1,\pi/4)}=0$$

$$\Rightarrow 1+\sec^2(\pi/4)\cdot\left(\left.\frac{dy}{dx}\right|_{(1,\pi/4)}-\pi/4\right)=0$$

$$\Rightarrow 1+2\cdot\left.\frac{dy}{dx}\right|_{(1,\pi/4)}-\frac{\pi}{4}\cdot 2=0\Rightarrow\left.\frac{dy}{dx}\right|_{(1,\pi/4)}=m_1=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

So, an eqn. of the tangent line is

$$y-\pi/4=\left(\frac{\pi-2}{4}\right)\cdot(x-1).$$

Since tang. and normal lines are perpendicular to each other,

$$m_1\cdot m_2=-1\Rightarrow m_2=\frac{4}{2-\pi}.$$

So, an eqn. of the normal line is

$$y-\pi/4=\frac{4}{2-\pi}(x-1)$$

(8) Show that the following functions are one-to-one and find $(f^{-1})'(1)$ and $(g^{-1})'(2\pi)$ where

(a) $f(x)=x^5+x+1$

(b) $g(x)=\sin x+x$

Solution: a) $f'(x)=5x^4+1>0$ for all $x\in\mathbb{R}$, so $f(x)$ is one-to-one. Therefore $f^{-1}(x)$ exists.

Recall that $\dfrac{d}{dx}f^{-1}(x_0)=\dfrac{1}{f'(f^{-1}(x_0))}$.

So, $(f^{-1})'(1)=\dfrac{1}{f'(f^{-1}(1))}=\dfrac{1}{f'(0)}=1$

$f^{-1}(1)=x \Rightarrow f(x)=1=x^5+x+1$ $\boxed{\Rightarrow x=0}$

$f'(x)=5x^4+1.$