MATH 119 — Recit. — Week 6

① Prove the following inequalities

(a) $\ln(1+x) < x$ for all $x > 0$

(b) $\cos x > 1 - \frac{x^2}{2}$ for all $x > 0$.

Solution: a) Let $x_0 > 0$ and $f(x) = \ln(1+x) - x$.

$f(x)$ is cont. on $[0, x_0]$ and diff. on $(0, x_0)$. So, by the MVT, there is some $c \in (0, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0}.$$

Then, $$f'(c) = \frac{1}{1+c} - 1 = \frac{-c}{1+c} = \frac{\ln(1+x_0) - x_0}{x_0} < 0$$

since $c > 0$. And since $x_0 > 0$, $\ln(1+x_0) - x_0 < 0$. So, for all $x > 0$, $\ln(1+x) < x$.

b) Let $f(x) = \cos x - 1 + \frac{x^2}{2}$ and $x_0 > 0$.

$f(x)$ is cont. on $[0, x_0]$ and diff. on $(0, x_0)$. So, by the MVT, there is some $c \in (0, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{\cos(x_0) - 1 + \frac{x_0^2}{2}}{x_0}$$

Note that $x_0 > 0$ implies if $f'(c) > 0$, then the proof is done. $f'(c) = -\sin(c) + c$. Here, in order to show that $-\sin(c) + c > 0$ for some $c \in (0, x_0)$, apply the MVT for $g(x) = x - \sin x$ for the interval $(0, x)$: ⊛ $g(x)$ is cont. on $[0, x]$ and diff. on $(0, x)$, so there is some $d \in (0, x)$ such that

$$1 - \cos d = g'(d) = \frac{g(x) - g(0)}{x - 0} = \frac{x - \sin x}{x} \geq 0.$$

If $c < 2\pi$, $c - \sin c = 0$ for only $c = 0$. So, for any $c > 0$, $c - \sin c > 0$.

(2) Find the intervals on which f is increasing and decreasing where $f(x) = xe^{-x}$.

Solution: $f'(x) = e^{-x} + x.(-1).e^{-x}$

$$= e^{-x}(1-x) = 0 \Rightarrow x = 1$$

		1	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

Since $f'(x) > 0$ for all $x \in (-\infty, 1)$, $f(x)$ is incr. on $(-\infty, 1)$

Since $f'(x) < 0$ for all $x \in (1, \infty)$, $f(x)$ is decr. on $(1, \infty)$

(3) Find $\frac{dy}{dx}$ at $x = 0$ if the diff. func. $y = y(x)$ is given implicitly by

$$3xe^{2y} + ye^x = 4e^x - 3$$

Solution: ⚠ Alway keep in mind that y is a function of x.

Take the derivatives of both sides with respect to x:

$$3e^{2y} + 3x.2y'e^{2y} + y'e^x + ye^x = 4e^x.$$

Now put $x = 0$,

$$3e^{2.y(0)} + y'(0) + y(0) = 4$$

$$\Rightarrow y'(0) = \left.\frac{dy}{dx}\right|_{x=0} = 4 - y(0) - 3e^{2y(0)}$$

In order to find $y(0)$, just $x = 0$ in the original equation:

$$y(0) = 4 - 3 = 1.$$

So, $y'(0) = 3 - 3e^2$

④ Show that $f(x) = x^5 + \arctan x + e^x + 119$ has an inverse func. (defined on the range of f) and find $(f^{-1})'(120)$.

Solution: $f'(x) = 5x^4 + \frac{1}{1+x^2} + e^x > 0$ for all $x \in \text{Dom}(f)$, so

$f(x)$ is incr. on its domain. Thus $f(x)$ has an inverse, i.e., it is invertible.

Recall that $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$.

$f^{-1}(120) = y \Rightarrow f(y) = 120 \Rightarrow y^5 + \arctan y + e^y + 119 = 120$

$\Rightarrow y = 0$.

So, $f^{-1}(120) = 0$. And $f'(x) = 5x^4 + \frac{1}{1+x^2} + e^x$. Thus,

$$(f^{-1})'(120) = \frac{1}{f'(0)} = \frac{1}{2}$$

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⑤ Find y' if

(a) $y = x^{a^2} + a^{x^2} + 2^{a^x}$ for some constant $a > 0$

(b) $y = \ln(\ln x) + e^{e^x}$

(c) $y = x^{\ln x}$

Solution: a) $y' = a^2 . x^{a^2 - 1} + 2x . a^{x^2} . \ln|a| + a^x . \ln a . 2^{a^x} . \ln 2$

b) $y' = \frac{\frac{1}{x}}{\ln x} + e^x . e^{e^x}$

c) $y = x^{\ln x} \Rightarrow \ln y = \ln(x^{\ln x}) \Rightarrow \ln y = \ln x . \ln x = (\ln x)^2$

$\Rightarrow \frac{y'}{y} = 2 . \ln x . \frac{1}{x} \Rightarrow y' = 2 \ln x . \frac{1}{x} . x^{\ln x}$

⑥ Find $\frac{dy}{dx}$ if

(a) $y = e^{-(\arcsin x)^2}$

(b) $y = 2x \arctan(x^2+1)$

(c) $y = \arctan(\arcsin\sqrt{x})$

Solution: a) $y' = (-2)\arcsin x \cdot \frac{1}{\sqrt{1-x^2}} \cdot e^{-(\arcsin x)^2}$

b) $y' = 2\arctan(x^2+1) + 2x \cdot \frac{1}{1+(x^2+1)^2} \cdot 2x$

c) $y' = \frac{1}{1+(\arcsin\sqrt{x})^2} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

⑦ Verify that the point $P_0(1, \frac{1}{2})$ is on the curve C defined implicitly by the equation

$$\arcsin(xy) = x^2y + 5x - 11/2 + \pi/6$$

Then find the line tangent to C at $P_0(1, 1/2)$.

Solution: Put $P_0(1, 1/2)$ in the equation:

$\arcsin(1 \cdot \frac{1}{2}) = \frac{1}{2} + 5 - \frac{11}{2} + \frac{\pi}{6} \Rightarrow \frac{\pi}{6} = \frac{\pi}{6}$, So P_0 is on the curve.

Now, take the derivatives of both sides with respect to x:

$\frac{1}{\sqrt{1-x^2y^2}} \cdot (y + xy') = 2xy + x^2y' + 5$, Put $P_0(1, \frac{1}{2})$:

$\Rightarrow \frac{2}{\sqrt{3}} \cdot (\frac{1}{2} + y'(1)) = 1 + y'(1) + 5 \Rightarrow y'(1) \cdot (\frac{2}{\sqrt{3}} - 1) = 6 - \frac{1}{\sqrt{3}}$

$\Rightarrow y'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \frac{6\sqrt{3}-1}{2-\sqrt{3}}$. So, an equation of the tangent line is

$$y - \frac{1}{2} = \frac{6\sqrt{3}-1}{2-\sqrt{3}}(x - 1/2)$$

(8) Show that the equation

$$4\arctan(x) = 3 - 2x - x^3$$

has exactly one solution on $(0,1)$.

Solution: Let $f(x) = 4\arctan(x) + x^3 + 2x - 3$.

$f(x)$ is cont. on $[0,1]$ since $4.\arctan(x)$ and the poly.. x^3+2x-3 are cont. on $[0,1]$.

$\left.\begin{array}{l} f(0) = -3 < 0 \\ f(1) = 4.\frac{\pi}{4} + 1+2-3 = \pi > 0 \end{array}\right\}$ So, by the I.V.T., there is some $c \in (0,1)$ such that $f(c) = 0$.

Now, on the contrary, assume that there is another root d, say $d > 0$.

which means, $f(d) = 0$. So,

$f(x)$ is cont. on $[c,d]$ and diff. on (c,d). So, by Rolle's thm, there is some $e \in (c,d)$ such that $f'(e) = 0$. Note that $f'(x) = \frac{4}{1+x^2} + 3x^2 + 2$

$$\Rightarrow \frac{4}{1+e^2} + 3e^2 + 2 = 0$$

$$\Rightarrow 3e^2 + 2 + 3e^4 + 2e^2 + 4 = 0 \Rightarrow 3e^4 + 5e^2 + 6 = 0$$

Let $e^2 = t$. Then, $3t^2 + 5t + 6 = 0$, but $\Delta = 25 - 4.3.6 < 0$,

So we get a contradiction.

Thus, there can not be another root.