Math 497 Hilbert Space Techniques
Fall 2019 MT I

1. (10pts) a) Find an inner product on the space $P[-1,1]$ of complex valued polynomials such that the norm is given by

$$||f|| = \left(\int_{-1}^{1} |x||f(x)|^2 + 3|f'(x)|^2 dx\right)^{1/2}.$$

b)(5pts) Prove that for any $f \in P[-1,1]$,

$$\left|\int_{-1}^{1} |x|^3 f(x) + 6xf'(x)dx\right| \leq \frac{5}{\sqrt{3}}\left(\int_{-1}^{1} |x||f(x)|^2 + 3|f'(x)|^2 dx\right)^{1/2}.$$

2. (10pts) a) Show that $(C[0,1], ||.||_\infty)$ is a Banach Space where $C[0,1]$ is the space of real valued continuous functions on $[0,1]$ and the sup norm is defined by $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. (You don't need to show that $||.||_\infty$ is a norm.)

b) (5pts) Show that $(C[0,1], ||.||_\infty)$ is not a Hilbert Space.

3. (15pts) Let R be the space of complex rational functions (quotients of polynomials) which have no poles in the closed unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. An inner product on R is defined by

$$< f, g >= \frac{1}{2\pi i}\int_{|z|=1} f(z)\overline{g(z)}\frac{dz}{z}.$$

Show that the orthogonal complement of $M^\perp$ of

$$M = \{z^n f(z) : f \in R\} \subset R$$

is the space of polynomials of degree less than n.

4. (15pts) Let y be a non-zero vector in a Hilbert space H and $M = \{x \in H :< x, y >= 0\}$. What is $M^\perp$?