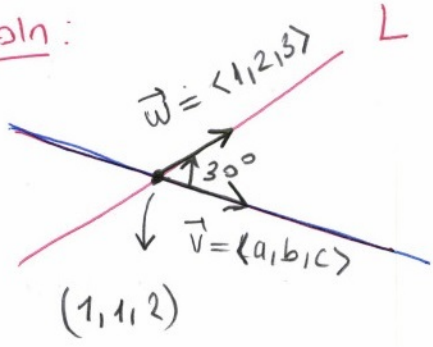
(3) Let $L: (x,y,z) = (1+t, 1+2t, 2+3t)$, $t \in \mathbb{R}$ be a given line.

What is the condition on $\vec{v} = \langle a,b,c \rangle$ s.t. the line $L_{\vec{v}} = (1,1,2) + t\vec{v}$ $t \in \mathbb{R}$ intersects the line L with a 30 angle. How many such lines $L_{\vec{v}}$ are there, and what is the union of all such lines $L_{\vec{v}}$ geometrically?

soln:



$L: (x,y,z) = (1+t, 1+2t, 2+3t)$, $t \in \mathbb{R}$

$\Rightarrow \vec{w} = \langle 1,2,3 \rangle$ is the direction vector of L.

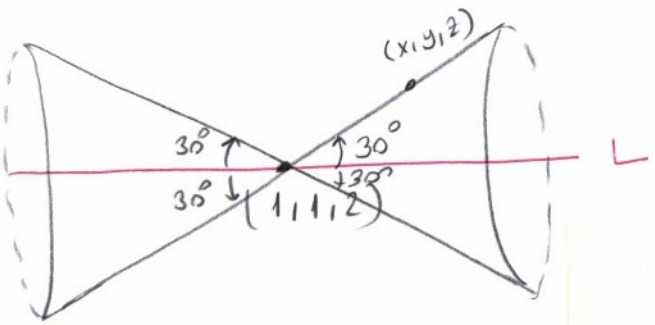
$L_{\vec{v}}$ intersects L with a 30 angle means that the angle between $\vec{w}$ and $\vec{v}$ is $\frac{\pi}{6}$. So, $\vec{w} \cdot \vec{v} = |\vec{w}||\vec{v}| \cos\theta = |\vec{w}| \cdot |\vec{v}| \cdot \cos\frac{\pi}{6}$

$$\Rightarrow \langle 1,2,3 \rangle \cdot \langle a,b,c \rangle = \sqrt{1^2+2^2+3^2}\sqrt{a^2+b^2+c^2}\,\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad a+2b+3c = \sqrt{14}\,\sqrt{a^2+b^2+c^2}\,\frac{\sqrt{3}}{2} \qquad (*)$$

So, if $(*)$ is satisfied, then $L_{\vec{v}}$ intersects L with a 30-angle.

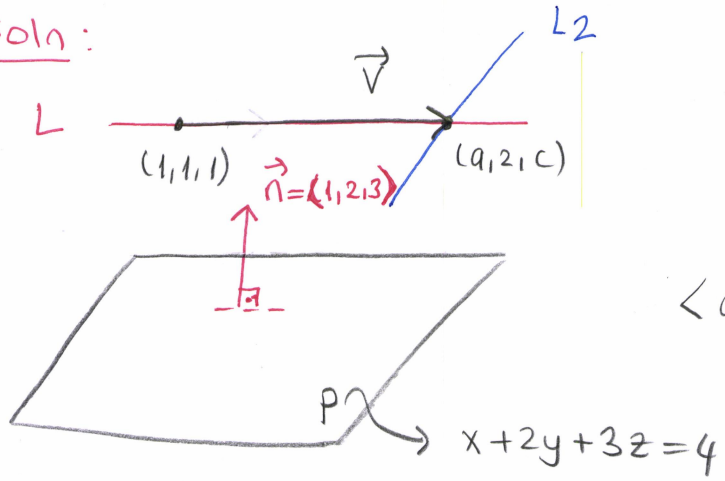
Infinitely many such lines $L_{\vec{v}}$ exist and union of these lines is "infinite cone".



$$\{(x,y,z)\in\mathbb{R}^3 : (x-1,\, y-1,\, z-2) = t(a,b,c),\ t\in\mathbb{R},\ \sqrt{14}\,\sqrt{a^2+b^2+c^2}\,\frac{\sqrt{3}}{2} = a+2b+3c\}$$

(5) Write down parametric eqns. of the line L which contains $(1,1,1)$ which is parallel to the plane $x+2y+3z=4$, and which intersects the line $L_2: x-1 = \frac{z-3}{3}, y=2$.

Soln:



Suppose $(a,2,c)$ is the intersection point of L_1 & L_2.

$\langle a-1, 2-1, c-1\rangle \perp \langle 1,2,3\rangle$

↳ direction vector of L

So, $\langle a-1, 1, c-1\rangle \cdot \langle 1,2,3\rangle = a-1+2+3c-3 = 0$

$\Rightarrow a+3c=2$

$(a,2,c)$ is on $L_2 \Rightarrow$ $a-1 = \frac{c-3}{3} \Rightarrow 3a=c$

$\Rightarrow a+3c=2 \Rightarrow a+9a=2 \Rightarrow 10a=2 \Rightarrow a=\frac{1}{5}$

$\Rightarrow c=\frac{3}{5}$

Also, $\langle a-1, 1, c-1\rangle = \langle -\frac{4}{5}, 1, -\frac{2}{5}\rangle$ is a direction vector for L. So,

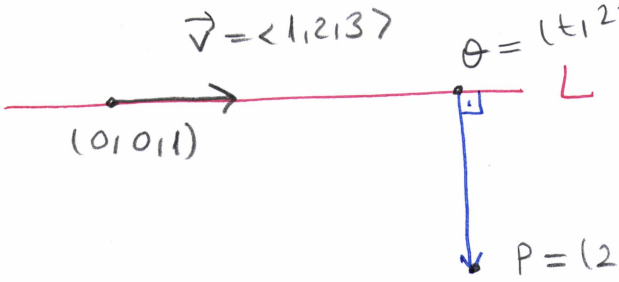
$$\left.\begin{aligned} x &= 1-\tfrac{4}{5}t \\ y &= 1+t \\ z &= 1-\tfrac{2}{5}t \end{aligned}\right\} \quad t\in\mathbb{R}$$

parametric eqns. for L.

(6) Find point $\theta \in L$ which is closest to point $P = (2,4,5)$ where L is the line $L: (x,y,z) = (t, 2t, 1+3t)$, $t \in \mathbb{R}$.

Calculate distance from P to L without using the distance formula between a point and a line.

Soln:



$(t, 2t, 1+3t) = (2,4,5)$

$\exists$ no such t.

So, P is not on L.

The direction vector of L is $\vec{v} = \langle 1,2,3 \rangle$.

$$\overrightarrow{\theta P} = \langle 2-t, 4-2t, 5-1-3t \rangle \perp \langle 1,2,3 \rangle$$

$$\text{So, } \langle 2-t, 4-2t, 4-3t \rangle \cdot \langle 1,2,3 \rangle = 0$$

$$\Rightarrow \quad 2-t+8-4t+12-9t = 0$$

$$22-14t = 0 \quad \Rightarrow \quad t = \frac{11}{7}$$

So, $\theta = \left(\frac{11}{7}, \frac{22}{7}, \frac{40}{7}\right) \in L$ is the closest point to P.

$$d(P,L) = |P\theta| = \sqrt{\left(2-\frac{11}{7}\right)^2 + \left(4-\frac{22}{7}\right)^2 + \left(5-\frac{40}{7}\right)^2}$$

$$= \sqrt{\frac{10}{7}}$$