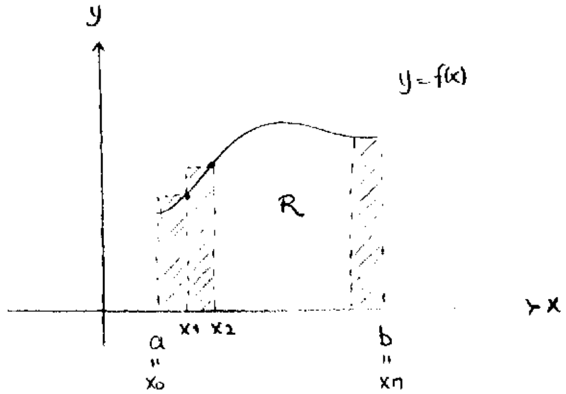
MATH 119 – CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 9 & 10

The Basic Area Problem: How to find the area of the region R lying under the graph of $y=f(x)$?



Divide $[a,b]$ into n subintervals:

$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

Let $\Delta x_i = x_i - x_{i-1}$ be the length of each subinterval.

For each subinterval build a rectangle whose base length is Δx_i and height is $f(x_i)$.
(x_i can be any point of $[x_{i-1}, x_i]$)

The approximate area of R is $\quad S_n = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \dots + f(x_n)\Delta x_n = \sum_{i=1}^{n} f(x_i)\Delta x_i$

And the exact area is $\quad \text{Area}(R) = \lim_{n\to\infty} S_n$

1. Interpret the given sum S_n as a sum of areas of rectangles approximating the area of a certain region in the plane and compute $\lim_{n\to\infty} S_n$ where

$$S_n = \sum_{i=1}^{n} \frac{i + n\sqrt{1-(i/n)^2}}{n^2}$$

Solution:

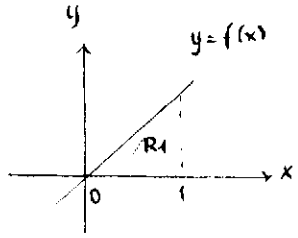
∇ In general, interval $[a,b]$ is divided into n-equal length subintervals and

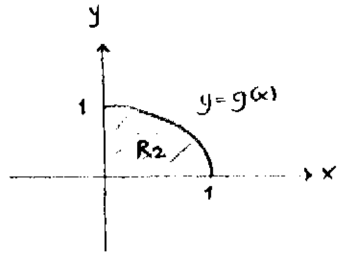
$\Delta x_i = \dfrac{b-a}{n}$, $\quad x_i = a + i\,\dfrac{b-a}{n}$

$$S_n = \sum_{i=1}^{n} \frac{i}{n^2} + \frac{1}{n}\sqrt{1-\left(\frac{i}{n}\right)^2} = \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x_i}\cdot\underbrace{\left(\frac{i}{n}\right)}_{f(x_i)} + \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x_i}\underbrace{\sqrt{1-\left(\frac{i}{n}\right)^2}}_{g(x_i)}$$

The interval is $[0,1]$ and $x_i = 0 + i\cdot\frac{1-0}{n} = \frac{i}{n}$ then $f(x) = x$ and $g(x) = \sqrt{1-x^2}$

Thus, S_n interprets the sum of the regions under the graphs of $f(x)$ and $g(x)$ over the interval $[0,1]$





$R_1 = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

$R_2 = \frac{1}{4}\pi\, 1^2 = \frac{\pi}{4}$

Therefore; $\quad \lim_{n\to\infty} S_n = R_1 + R_2 = \frac{1}{2} + \frac{\pi}{4}$

The Riemann Sum: Let f be a continuous and non-negative function on $[a,b]$.

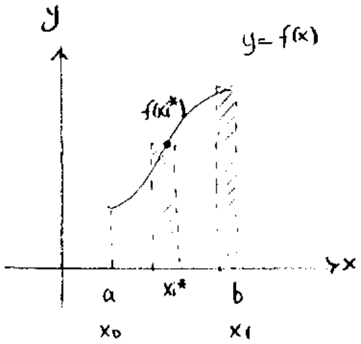
The finite set of points $P = \{x_0, x_1, x_2, \dots x_n\}$ such that $a = x_0 < x_1 < x_2 < \dots < x_n = b$ defines a partition for $[a,b]$ and divides $[a,b]$ into n-subintervals $[x_{i-1}, x_i]$.

The length of each sub-interval is $\Delta x_i = x_i - x_{i-1}$.

Let $x_i^* \in [x_{i-1}, x_i]$ be an arbitraey element of i^{th} subinterval.

The Riemann sum of f for the partition P is

$$R(f,P) = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \dots + f(x_n^*)\,\Delta x_n = \sum_{i=1}^{n} f(x_i^*).\Delta x_i$$



Let $f(l_i)$; $l_i \in [x_{i-1}, x_i]$ be the absolute minimum value of f on i^{th} subinterval, the lower Riemann sum of f for P is

$$L(f,P) = f(l_1)\,\Delta x_1 + \dots + f(l_n)\,\Delta x_n = \sum_{i=1}^{n} f(l_i)\,\Delta x_i$$

Let $f(u_i)$; $u_i \in [x_{i-1}, x_i]$ be the absolute maximum value of f on i^{th} subinterval, the upper Riemann sum of f foe P is

$$U(f,P) = f(u_1)\,\Delta x_1 + \dots + f(u_n)\,\Delta x_n = \sum_{i=1}^{n} f(u_i)\,\Delta x_i$$

The Definite Integral: Suppose there is exactly one I such that for eveey partition P of $[a,b]$ we have $L(f,P) \leq I \leq U(f,P)$ then f is integrable on $[a,b]$ and the definite integral of f is

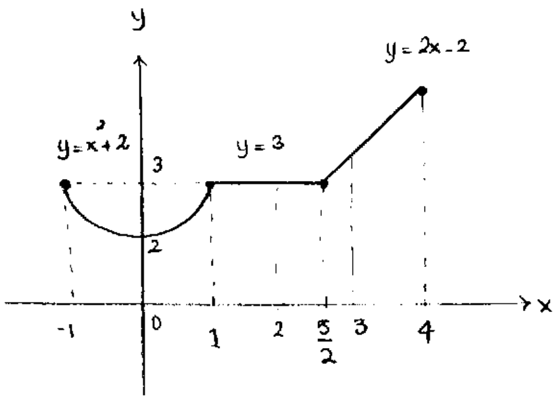
$$I = \int_a^b f(x)\,dx.$$

∇ If f is integrable, for any partition P, $\lim_{n \to \infty} R(f,P_n) = \int_a^b f(x)\,dx$

2. Consider the function $f(x) = \begin{cases} x^2+2 & ; \text{ if } x \leqslant 1 \\ 3 & \text{ if } 1 < x \leqslant 5/2 \\ \text{[illegible]} & \text{ if } x > 5/2 \end{cases}$ over the interval $[-1, 4]$.

Compute $L(f, P_5)$ and $U(f,$ [illegible] P_5 is the p[illegible]on of $[-1,4]$ into 5-subintervals of equal length. What can you say about the values of $L(f, P_4)$ and $U(f, P_4)$ when you compare with $L(f, P_5)$ and $U(f, P_5)$?

Solution:



P_5 is the partition of $[-1,4]$ into 5-subintervals of equal length:

$\Delta x_i = \frac{4-(-1)}{5} = 1$

$P_5 = \{-1, 0, 1, 2, 3, 4\}$

The lower Riemann Sum:

On the given sub-intervals minimum value of f:

$[-1,0]$: $f(0) = 0+2 = 2$

$[0,1]$: $f(0) = 0+2 = 2$

$[1,2]$: $f(2) = 3$ (constant)

$[2,3]$: $f(5/2) = 3$

$[3,4]$: $f(3) = 6-2 = 4$

$$L(f, P_5) = \sum_{i=1}^{5} f(l_i)\,\Delta x_i = f(0)\,\Delta x_1 + f(0)\,\Delta x_2 + f(2)\,\Delta x_3 + f(5/2)\,\Delta x_4 + f(3)\,\Delta x_5$$
$$= 2+2+3+3+4 = 14$$

The Upper Riemann Sum:

On the given sub-intervals maximum value of f:

$[-1,0]$: $f(-1) = 1+2 = 3$

$[0,1]$: $f(1) = 1+2 = 3$

$[1,2]$: $f(2) = 3$

$[2,3]$: $f(3) = 6-2 = 4$

$[3,4]$: $f(4) = 8-2 = 6$

$$U(f, P_5) = \sum_{i=1}^{5} f(u_i)\,\Delta x_i = f(-1)\,\Delta x_1 + f(1)\,\Delta x_2 + f(2)\,\Delta x_3 + f(3)\,\Delta x_4 + f(4)\,\Delta x_5$$
$$= 3+3+3+4+6 = 19$$

3. Write a Riemann sum with equal length sub-intervals to compute the integral $\int_{-2}^{1} \left[x^2 + x \cdot \sin\left(\frac{1}{x+3}\right) \right] dx$.

Solution:

Let P_n divide $[-2,1]$ into n equal length sub-intervals.

$\Delta x_i = \frac{1-(-2)}{n} = \frac{3}{n}$ and for x_i^* choose Right end points of each sub-interval; that is,

$x_i^* = -2 + i \cdot \frac{3}{n}$

Then the Riemann sum of $f(x) = x^2 + x \cdot \sin\left(\frac{1}{x+3}\right)$ for the partition P_n

$$R(f, P_n) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i = \sum_{i=1}^{n} \left[\left(-2 + i \cdot \frac{3}{n}\right)^2 + \left(-2 + i \cdot \frac{3}{n}\right) \sin\left(\frac{1}{(-2 + i \cdot \frac{3}{n}) + 3}\right) \right] \frac{3}{n}$$

$\lim_{n \to \infty} R(f, P_n)$ gives the result of the definite integral

4. Express the following limits as definite integrals.

(a) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \arctan\left(3 + \frac{2i}{n}\right)$

(b) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \sin\left(\frac{2i-n}{2n}\right)$

(c) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n+2i}$

Solution:

(a) $$\lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x_i} \cdot \underbrace{\arctan\left(3 + \frac{2i}{n}\right)}_{f(x_i^*)}$$

$\Delta x_i = \frac{1}{n}$ choose the interval as $[0,1]$ and Right end points of sub-intervals can be expressed as $x_i^* = 0 + i \cdot \frac{1}{n} = \frac{i}{n}$

Choose $f(x) = \arctan(3 + 2x)$ that is integrable so this limit can be expressed as

$$\int_0^1 \arctan(3 + 2x)\, dx.$$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x_i} \cdot \underbrace{\sin\left(\frac{2i}{2n} - \frac{n}{2n}\right)}_{f(x_i^*)}$

$\Delta x_i = \frac{1}{n}$ choose the interval as $[0,1]$ and right end points of sub-intervals can be expressed as $x_i^* = 0 + i \cdot \frac{1}{n} = \frac{i}{n}$

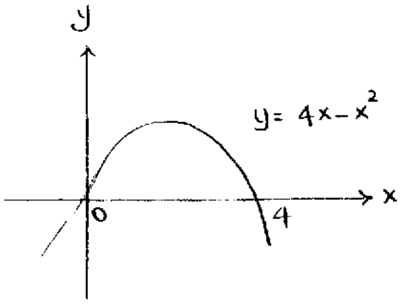
Choose $f(x) = \sin(x - 1/2)$ that is integrable so this limit can be expressed as

$$\int_0^1 \sin(x - 1/2)\, dx.$$

5. Find the values of a and b which maximize the integral $\int_a^b (4x - x^2)\, dx$

Solution:

The definite integral of f over $[a,b]$ gives the area of the region bounded by f and x-axis over $[a,b]$ if f is positive valued. (For negative values of f result is negative)



To maximize the result of definite integral take just positive part of f

So $\int_0^4 (4x - x^2)\, dx$ is maximum value.

6. Using a Riemann sum show that $\int_a^b x^2 dx = b^3 - a^3$

Solution:

let P_n be partition of $[a,b]$, divides $[a,b]$ into n-equal length subintervals.

$\Delta x_i = \frac{b-a}{n}$ and let $x_i^* = a + i\,\frac{b-a}{n}$ right end points of each sub-interval.

The Riemann sum of $f(x) = x^2$

$$R(f, P_n) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i = \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right) \cdot \frac{b-a}{n}$$

$$= \sum_{i=1}^{n} \left(a + \frac{i(b-a)}{n}\right)^2 \cdot \frac{b-a}{n}$$

$$R(f,P_n) = \sum_{i=1}^{n}\left(a + i\frac{(b-a)}{n}\right)^2\frac{(b-a)}{n} = \sum_{i=1}^{n}\left(a^2 + \frac{2a(b-a)i}{n} + \frac{i^2(b^2-2ab+a^2)}{n^2}\right)\frac{(b-a)}{n}$$

$$= \sum_{i=1}^{n}\left[\frac{a^2}{n} + \frac{2a(b-a)}{n^2}\, i + \frac{(b^2-2ab+a^2)}{n^3}\, i^2\right] \cdot (b-a)$$

$$= \left[\frac{a^2}{\cancel{n}}\cdot\cancel{n} + \frac{\cancel{2}a(b-a)}{n^2}\,\frac{n(n+1)}{\cancel{2}} + \frac{b^2-2ab+a^2}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right](b-a)$$

$$\lim_{n\to\infty} R(f,P_n) = \left[\cancel{a^2} + ab - \cancel{a^2} + \frac{b^2-2ab+a^2}{3}\right](b-a)$$

$$= \left[\frac{3ab+b^2-2ab+a^2}{3}\right](b-a)$$

$$= \frac{1}{3}(b-a)(a^2+ab+b^2)$$

$$= \frac{1}{3}\left(b^3-a^3\right)$$

Since $f(x)=x^2$ is continuous, it is integrable :

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R(f,P_n)$$

$$\int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3)$$

The Fundamental Theorem of Calculus: Suppose that f is continuous on I containing a.

(i) Let F be defined by $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on I and $F'(x) = f(x)$.

F is an anti-derivative of f.

(ii) If G is any anti-derivative of f; $\int_a^b f(x)\,dx = G(b) - G(a)$.

∇_0 $\quad \dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)\,dt = f(g(x)).g'(x)$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x).$$

7. Find $f'(x)$ for

(a) $f(x) = \displaystyle\int_{-2}^{\sin x} \frac{1}{\sqrt{1+t^4}}\,dt$

(b) $f(x) = \displaystyle\int_{e^x}^{\tan^2 x} \sin(t^2)\,dt$

Solution:

(a) $f'(x) = \dfrac{1}{\sqrt{1+\sin^4 x}} \,.\, \cos x$

(b) $f'(x) = \sin(\tan^4 x)\,.\,2\tan x\,(1+\tan^2 x) - \sin(e^{2x})\,.\,e^x$

The Method of Substitution: $I = \int f'(g(x))\,g'(x)\,dx$

Let $u = g(x)$ then $du = g'(x).dx$

Thus; $I = \int f'(u)du = f(u) + C = f(g(x)) + C$

∇_0 For definite integrals $\displaystyle\int_a^b f(g(x))\,g'(x)\,dx = \int_A^B f(u)\,du$ where $A = g(a)$, $B = g(b)$

8. Evaluate the following indefinite integrals:

(a) $f(x) = \int x^2 . e^{4+x^3} dx$

Solution:

let $u = 4 + x^3 \Rightarrow du = 3x^2 dx$

$f(x) = \frac{1}{3}\int 3x^2 . e^{4+x^3} dx = \frac{1}{3}\int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{4+x^3} + C.$

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RECITATION 11

The Method of Substitution: $I = \int f'(g(x))\, g'(x)\, dx$

Let $u = g(x)$ then $du = g'(x)dx$

Thus, $I = \int f'(u)\,du = f(u) + C = f(g(x)) + C$

For definite integrals; $\int_a^b f(g(x))\, g'(x)\, dx = \int_A^B f(u)\,du$ where $A = g(a)$ and $B = g(b)$.

1. Evaluate the indefinite integrals.

(a) $f(x) = \int x^2 e^{4+x^3}\, dx$

(b) $f(x) = \int \frac{x - \sin x}{x^2 + 2\cos x}\, dx$

(c) $f(x) = \int \frac{x^2+1}{\sqrt{x^3+3x-2}}\, dx$

(d) $f(x) = \int 3^x\, dx$

(e) $f(x) = \int e^x \sqrt{1+4e^x}\, dx$

Solution:

(a) $\int x^2 . e^{4+x^3}\, dx = \frac{1}{3}\int e^{4+x^3}\, 3x^2 dx = \frac{1}{3}\int e^u\, du = \frac{1}{3}e^u + C = \frac{1}{3}e^{4+x^3} + C$

Let $u = 4 + x^3$

$du = 3x^2 dx$

(b) $\int \frac{x - \sin x}{x^2 + 2\cos x}\, dx = \frac{1}{2}\int \frac{1}{x^2+2\cos x}\, 2(x - \sin x)dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2\cos x| + C$

Let $u = x^2 + 2\cos x$

$du = (2x - 2\sin x)\,dx$

(c) $\int \frac{x^2+1}{\sqrt{x^3+3x-2}}\, dx = \frac{1}{3}\int (x^3+3x-2)^{-1/2}\, 3(x^2+1)dx = \frac{1}{3}\int u^{-1/2} du = \frac{1}{3}\cdot\frac{u^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}\sqrt{x^3+3x-2} + C$

Let $u = x^3 + 3x - 2$

$du = (3x^2+3)dx$

(d) $\int 3^x\, dx = \frac{3^x}{\ln 3} + C$

2. Evaluate the following integrals

(a) $\int \sin^5 x \cdot \cos x \, dx$

(b) $\int \sin^2 x \cdot \cos^2 x \, dx$

(c) $\int \sec^6 x \cdot \tan^2 x \, dx$

(d) $\int \sec^3 x \cdot \tan^3 x \, dx$

Solution:

(a) $\int \sin^5 x \cdot \cos x \, dx = \int u^5 du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C$

let $u = \sin x$

$du = \cos x \, dx$

(b) $\int \sin^2 x \cdot \cos^2 x \, dx = \frac{1}{4} \int 4 \sin^2 x \cdot \cos^2 x \, dx = \frac{1}{4} \int (\sin 2x)^2 dx = \frac{1}{4} \int \frac{1 - \cos 4x}{2} dx$

$= \frac{1}{8} \left[x - \frac{\sin 4x}{4} \right] + C$

Recall that $\sin 2x = 2 \sin x \cdot \cos x$

$\cos 4x = 1 - 2 \sin^2 2x \Rightarrow \sin^2 2x = \frac{1 - \cos 4x}{2}$

(c) Observe that $\sec^2 x = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x$

$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\int \sec^6 x \cdot \tan^2 x \, dx = \int \sec^4 x \cdot \tan^2 x \cdot \sec^2 x \, dx = \int (1 + \tan^2 x)^2 \cdot \tan^2 x \sec^2 x dx$

let $u = \tan x$

$du = \sec^2 x dx$

$= \int (1+u^2)^2 u^2 du = \int (1 + 2u^2 + u^4) u^2 du$

$= \int (u^2 + 2u^4 + u^6) du = \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C$

$= \frac{\tan^3 x}{3} + \frac{2 \tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

(d) Exercise.

Integration by Parts: Suppose that $u(x)$ and $v(x)$ are two differentiable functions.

$$\frac{d}{dx}(u(x).v(x)) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx} \Rightarrow \int u\frac{dv}{dx} = u.v - \int v\frac{du}{dx}$$

$$\int u\,dv = u.v - \int v.du$$

3. Evaluate the following integrals.

(a) $\int (x+3)e^{2x}dx$

(b) $\int_1^e \sin(\ln x)\,dx$

(c) $\int x\sin^2 x\,dx$

Solution:

(a) $\int (x+3)e^{2x}dx = (x+3).\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}dx = (x+3).\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$

$u = x+3 \qquad dv = e^{2x}dx$

$du = dx \qquad v = \frac{e^{2x}}{2}$

(b) $I = \int_1^e \sin(\ln x)\,dx = \sin(\ln x).x\Big|_1^e - \int_1^e x\cos(\ln x).\frac{1}{x}dx = e.\sin 1 - \int_1^e \cos(\ln x)\,dx$

$u = \sin(\ln x) \qquad dv = dx$

$du = \cos(\ln x)\frac{1}{x}dx \qquad v = x$

$u = \cos(\ln x) \qquad dv = dx$

$du = -\sin(\ln x)\frac{1}{x}dx \qquad v = x$

$$= e.\sin 1 - \left[\cos(\ln x).x\Big|_1^e - \int_1^e x\left(-\sin(\ln x)\frac{1}{x}\right)dx\right]$$

$$I = e.\sin 1 - e.\cos 1 + 1 - I \Rightarrow I = \frac{1}{2}\left[e.\sin 1 - e\cos 1 + 1\right]$$

(c) $\int x.\sin^2 x\,dx = \frac{1}{2}\int x(1-\cos 2x)dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x.\cos 2x\,dx$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$u = x \qquad dv = \cos 2x$

$du = dx \qquad v = \frac{\sin 2x}{2}$

$$= \frac{1}{2}.\frac{x^2}{2} - \frac{1}{2}\left[x.\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}dx\right] = \frac{1}{4}.x^2 - \frac{1}{4}x.\sin 2x - \frac{1}{8}\cos 2x + C.$$

4. Evaluate the following integrals

(a) $\int \frac{x^2-8}{x^2-9}\,dx$

(b) $\int \frac{3x}{(x-1)^2(x^2+x+1)}\,dx$

Solution:

(a) $\frac{x^2-8}{x^2-9} = \frac{x^2-9+1}{x^2-9} = 1 + \frac{1}{x^2-9}$

$\frac{1}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$
$(x-3) \quad (x+3)$

$Ax - 3A + Bx + 3B = 1 \Rightarrow$ $A+B = 0 \Rightarrow A = -B$

$-3A+3B = 1 \Rightarrow 6B = 1 \Rightarrow B = \frac{1}{6} \quad A = -\frac{1}{6}$

$$\int \frac{x^2-8}{x^2-9}\,dx = \int \left[1 - \frac{1}{6}\cdot\frac{1}{x+3} + \frac{1}{6}\,\frac{1}{x-3} \right] dx = x - \frac{1}{6}\cdot \ln|x+3| + \frac{1}{6}\ln|x-3| + C$$

(b) $\frac{3x}{(x-1)^2(x^2+x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} \Rightarrow$ $A=0$, $B=1$, $C=0$, $D=-1$

$$I = \int \frac{3x}{(x-1)^2(x^2+x+1)}\,dx = \int \left[\frac{1}{(x-1)^2} - \frac{1}{x^2+x+1} \right] dx$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx = \int \frac{1}{\frac{3}{4}\left[1 + \frac{(x+1/2)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}\right]}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{(x+1/2)\cdot 2}{\sqrt{3}}\right)^2}\,dx$$

$u = (x+1/2)\cdot\frac{2}{\sqrt{3}} \Rightarrow du = \frac{2}{\sqrt{3}}\,dx$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{1+u^2}\,du = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

$$I = \frac{(x-1)^{-1}}{(-1)} + \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

The Inverse Trigonometric Substitutions

* Integrals involving $\sqrt{a^2-x^2}$ $(a>0)$ [illegible] reduced [illegible] form by means of the substitution $x = a.\sin\theta$.
* $\sqrt{a^2+x^2}$ or $\dfrac{1}{x^2+a^2}$ $\longrightarrow$ $x = a.\tan\theta$
* $\sqrt{x^2-a^2}$ $\longrightarrow$ $x = a.\sec\theta$

5. Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$

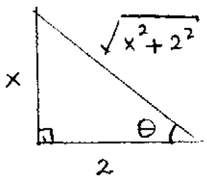
(b) $\displaystyle\int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx$

Solution:

(a) Integral involves x^2+2^2, use $x = 2.\tan\theta$ substitution.

$\tan\theta = \dfrac{x}{2}$

$dx = 2\,(1+\tan^2\theta)\,d\theta$



$$\int \frac{1}{(4\tan^2\theta+4)^2}\,2(1+\tan^2\theta)d\theta = \frac{2}{16}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{16}\int (\cos 2\theta+1)d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{1}{16}\left[\frac{\sin 2\theta}{2}+\theta\right]+C = \frac{1}{32}\,2\sin\theta.\cos\theta + \arctan\frac{x}{2}+C$$

$\cos^2\theta = \dfrac{\cos 2\theta+1}{2}$

$$= \frac{1}{16}\;\frac{2x}{x^2+4} + \arctan\frac{x}{2}+C.$$