MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 12

Improper Integrals: Let $I = \int_a^b f(x)dx$

(i) I is called improper integral TYPE 1 if $a = -\infty$ OR $b = \infty$ OR both.

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx.$$

(ii) I is called improper integral TYPE 2 if f is unbounded when $x \to a^+$ OR $x \to b^-$ OR both.

If $\lim_{x\to a^+} f(x) = \infty$ then $\int_a^b f(x)dx = \lim_{R\to a^+} \int_R^b f(x)dx.$

1. Evaluate the given integral OR show that it diverges.

(a) $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

(b) $\int_0^1 \frac{x+1}{x^2+2x}dx$

(c) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}dx$

(d) $\int_1^\infty \frac{1}{x(\ln x)^2}dx$

(e) $\int_{-\infty}^\infty \frac{x^4}{1+x^{10}}dx$

Solution:

(a) $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ is an improper integral TYPE 1.

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{-\sqrt{x}} \frac{1}{2\sqrt{x}}dx = 2\int e^{-u}du = -2e^{-u} + C = -2e^{-\sqrt{x}} + C$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{R\to\infty}\int_1^R \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{R\to\infty}\left[-2\cdot e^{-\sqrt{x}}\right]\Big|_1^R = -2\lim_{R\to\infty}\left[e^{-\sqrt{R}} - e^{-1}\right]$$

$$= -2\lim_{R\to\infty}\left[\overbrace{\frac{1}{e^{\sqrt{R}}}}^{0} - \frac{1}{e}\right] = \frac{2}{e}$$

The integral converges to $\frac{2}{e}$.

(b) $\int_0^1 \frac{x+1}{x^2+2x}\,dx$ is improper integral TYPE 2 since $\lim_{x\to 0^+} \frac{x+1}{x^2+2x} = \infty$

$$\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+c = \frac{1}{2}\ln|x^2+2x|+c$$

$u = x^2+2x$
$du = (2x+2)dx$

$$\int_0^1 \frac{x+1}{x^2+2x}\,dx = \lim_{R\to 0^+}\int_R^1 \frac{x+1}{x^2+2x}\,dx = \lim_{R\to 0^+}\left[\frac{1}{2}\ln|x^2+2x|\right]\Big|_R^1$$

$$= \lim_{R\to 0^+} \frac{1}{2}\left[\ln 3 - \ln|R^2+2R|\right] = \infty$$

The integral diverges to ∞.

(c) $\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$ is improper integral.

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^c \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_c^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx \quad , \quad c\in(0,\infty)$$

$$\int \frac{1}{\sqrt{x}\,(1+x)}\,dx = 2\int \frac{1}{(1+x)}\,\frac{1}{2\sqrt{x}}\,dx = 2\int \frac{1}{1+u^2}\,du = 2\arctan u + c = 2\arctan(\sqrt{x}) + c.$$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{R1\to 0}\int_{R1}^{c} \frac{1}{\sqrt{x}(1+x)}\,dx + \lim_{R2\to\infty}\int_c^{R2} \frac{1}{\sqrt{x}(1+x)}\,dx$$

$$= \lim_{R1\to 0}\left[2\arctan(\sqrt{x})\right]\Big|_{R1}^{c} + \lim_{R2\to\infty}\left[2\arctan(\sqrt{x})\right]\Big|_{c}^{R2}$$

$$= \lim_{R1\to 0}\left[2\left(\arctan\sqrt{c} - \arctan\sqrt{R1}\right)\right] + \lim_{R2\to\infty}\left[2\left(\arctan(\sqrt{R2}) - \arctan\sqrt{c}\right)\right]$$

$$= \left(\cancel{2\arctan\sqrt{c}} - 0\right) + \left(2\cdot\frac{\pi}{2} - \cancel{2\arctan\sqrt{c}}\right) = \pi$$

<u>P-Integrals:</u> Let $0<a<\infty$

(i) $\displaystyle\int_a^\infty \frac{1}{x^P}\,dx = \begin{cases} \text{Converges to } \dfrac{a^{1-P}}{P-1} & \text{if } P>1 \\ \text{Diverges to } \infty & \text{if } P\leq 1 \end{cases}$

(ii) $\displaystyle\int_0^a \frac{1}{x^P}\,dx = \begin{cases} \text{Converges to } \dfrac{a^{1-P}}{1-P} & \text{if } P<1 \\ \text{Diverges to } \infty & \text{if } P\geq 1 \end{cases}$

<u>Comparison Theorem:</u> Let $-\infty \leq a < b \leq \infty$ and f & g be continuous on (a,b). Assume that $0 \leq f(x) \leq g(x)$.

(i) If $\int_a^b g(x)\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is convergent.

(ii) If $\int_a^b f(x)\,dx$ is divergent, then $\int_a^b g(x)\,dx$ is divergent.

<u>Absolute Convergence:</u> If $\int_a^b |f(x)|\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is absolutely convergent. If an integral is absolutely convergent, then it is convergent.

<u>Limit Comparison Test:</u> Let $f(x) \geq 0$ and $g(x) \geq 0$. If the singularities of f and g on $[a,b]$ is at b and

$$\lim_{x\to b} \frac{f(x)}{g(x)} = \begin{cases} 0 \longrightarrow g(x) \gg f(x) \to \text{Comparison Test} \\ \infty \longrightarrow f(x) \gg g(x) \to \text{Comparison Test} \\ 0<L<\infty \longrightarrow \int_a^b f(x)\,dx \text{ and } \int_a^b g(x)\,dx \text{ have the same behaviour.} \end{cases}$$

2. Determine whether the given integral is convergent or divergent.

(a) $\displaystyle\int_1^\infty \frac{x}{x^3+\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{x^{5/2}+x^{7/3}}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{\arcsin x}\,dx$

(d) $\displaystyle\int_3^\infty \frac{1}{\sqrt{x}\,\ln x}\,dx$

(e) $\displaystyle\int_{10}^\infty \frac{\ln x}{x^2+e}\,dx$

Solution:

(a) $\displaystyle\int_1^\infty \frac{x}{x^3+\sqrt{x}}\,dx$

Observe that $0 \leqslant \dfrac{x}{x^3+\sqrt{x}} \leqslant \dfrac{x}{x^3} = \dfrac{1}{x^2}$ for $x \in [1, \infty)$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by P-Test, by Comparison Test $\displaystyle\int_1^\infty \frac{x}{x^3+\sqrt{x}}\,dx$ is also convergent.

(b) $\displaystyle\int_0^1 \frac{1}{x^{5/2}+x^{7/3}}\,dx$, let $f(x) = \dfrac{1}{x^{5/2}+x^{7/3}}$. f is unbounded as $x \to 0^+$

Define $g(x) = \dfrac{1}{x^{14/6}}$, g is also unbounded as $x \to 0^+$.

For $x \in (0,1)$, $f(x) > 0$ and $g(x) > 0$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x^{5/2}+x^{7/3}}}{\frac{1}{x^{14/6}}} = \lim_{x\to 0^+} \frac{\cancel{x^{14/6}}}{\cancel{x^{14/6}}(x^{1/6}+1)} = 1$$ thus $\int f$ & $\int g$ both converge or both diverge

$\displaystyle\int_0^1 \frac{1}{x^{14/6}}\,dx$ is divergent by P-test $(14/6 > 1)$, then by the Limit Comparison Test

$\displaystyle\int_0^1 \frac{1}{x^{5/2}+x^{7/3}}\,dx$ is divergent.

(c) $\displaystyle\int_0^1 \frac{1}{\arcsin x}\,dx$. let $f(x) = \dfrac{1}{\arcsin x}$ and $g(x) = \dfrac{1}{x}$, f & g are unbounded as $x \to 0^+$

and for $x \in (0,1)$ $f(x) > 0$ and $g(x) > 0$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{1}{\arcsin x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{\arcsin x} \underset{\left[\frac{0}{0}\right]}{\overset{L'H}{=}} \lim_{x\to 0^+} \frac{1}{\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 0^+} \sqrt{1-x^2} = 1$$

thus $\int f$ & $\int g$ both converge or both diverge.

$\displaystyle\int_0^1 \frac{1}{x}\,dx$ is divergent by P-Test, by Limit Comparison Test $\displaystyle\int_0^1 \frac{1}{\arcsin x}\,dx$ is also divergent.

(d) $\displaystyle\int_3^{\infty} \frac{1}{\sqrt{x}\,\ln x}\,dx$

$f(x) = \dfrac{1}{x^{1/2}\ln x}$, $g(x) = \dfrac{1}{x^{3/4}}$. FOR $x \in (3,\infty)$ $f(x) > 0$, $g(x) > 0$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^{1/2}\ln x}{1/x^{3/4}} = \lim_{x\to\infty} \frac{x^{3/4}}{x^{2/4}\cdot\ln x} = \lim_{x\to\infty} \frac{x^{1/4}}{\ln x} \quad \left[\tfrac{\infty}{\infty}\right] \ \text{L'H}$$

$$= \lim_{x\to\infty} \frac{1/4 \cdot x^{-3/4}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{4}\cdot x^{1/4} = \infty$$

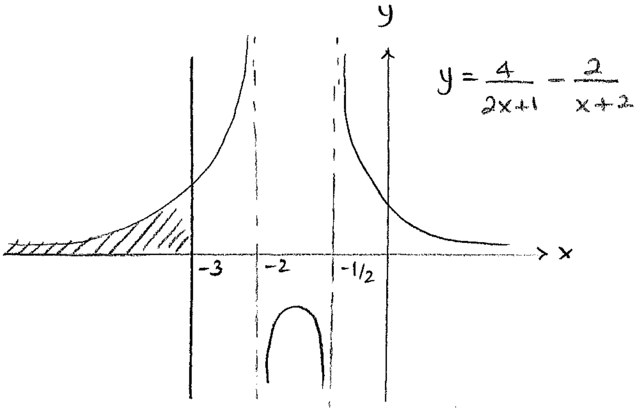
Thus; we have $g(x) \ll f(x)$.

$\displaystyle\int_3^{\infty} \frac{1}{x^{3/4}}\,dx$ is divergent by P-Test, by Limit Comparison Test $\displaystyle\int_3^{\infty} \frac{1}{\sqrt{x}\ln x}\,dx$

is also divergent.

3. Find the area of the region above the x-axis, to the left of the vertical line $x=-3$ and below the graph of $y = \frac{4}{2x+1} - \frac{2}{x+2}$.

<u>Solution;</u>



$$\text{Area} = \int_{-\infty}^{-3} \left(\frac{4}{2x+1} - \frac{2}{x+2} \right) dx$$

$$= \lim_{R \to \infty} \int_{R}^{-3} \left(\frac{4}{2x+1} - \frac{2}{x+2} \right) dx$$

$$= \lim_{R \to \infty} \left[\frac{4}{2} \ln|2x+1| - 2\ln|x+2| \right] \Bigg|_{R}^{-3}$$

$$= \lim_{R \to \infty} \left[2\ln 5 - \overset{0}{\cancel{2\ln 1}} - 2\ln\left|\frac{2R+1}{R+2}\right| \right]$$

$$= 2\ln 5 - 2\ln 2$$

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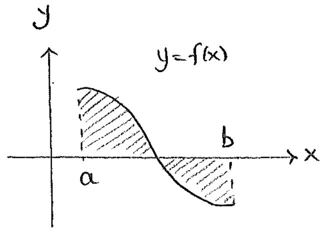
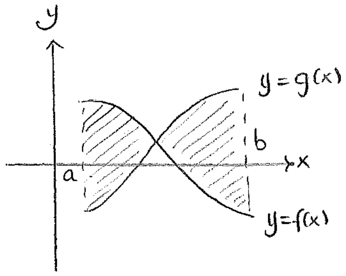
RECITATION 13

Areas of Plane Regions: The area between the graph of f and the x-axis from $x=a$ to $x=b$:



$$\text{Area} = \int_a^b |f(x)|\,dx$$

The area between the curves $y=f(x)$ and $y=g(x)$ from $x=a$ to $x=b$:

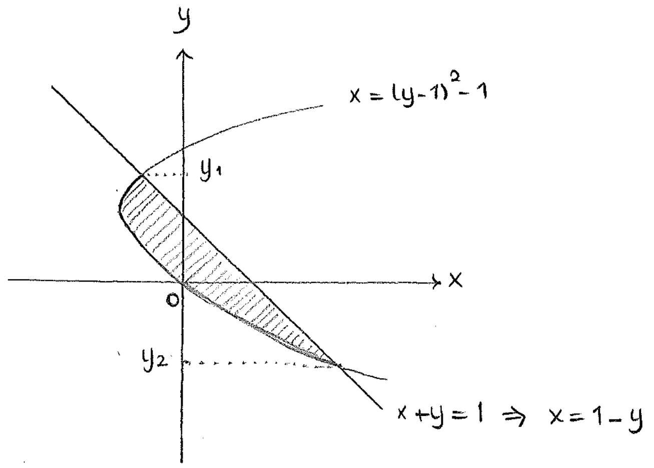


$$\text{Area} = \int_a^b |f(x)-g(x)|\,dx$$

1. Find the area of the indicated region
   (a) Bounded by the parabola $x=(y-1)^2-1$ and the line $x+y=1$.
   (b) Bounded by the parabolas $y=2x^2$ and $y=-x^2+3x+6$.

Solution:

(a)



Intersection of $x=(y-1)^2-1$ and $x+y=1$

$x+y=1 \Rightarrow x=1-y$

$1-y = y^2-2y+\cancel{1}-\cancel{1} \Rightarrow y^2-y-1=0$

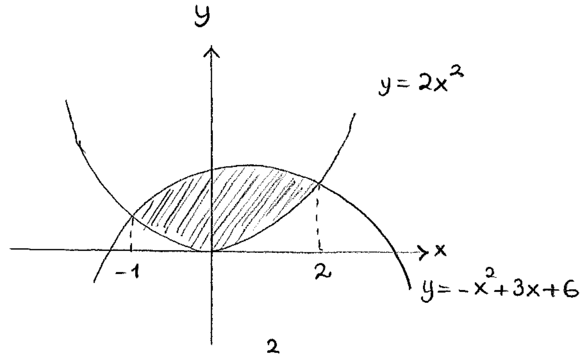
$\Delta = 1-4(1).(-1)=5$

$y_1 = \dfrac{1+\sqrt{5}}{2} \qquad y_2 = \dfrac{1-\sqrt{5}}{2}$

The area is

$$A = \int_{y_2}^{y_1} \left((1-y)-[(y-1)^2-1]\right) dy$$

(b)



Intersection of $y=2x^2$ and $y=-x^2+3x+6$

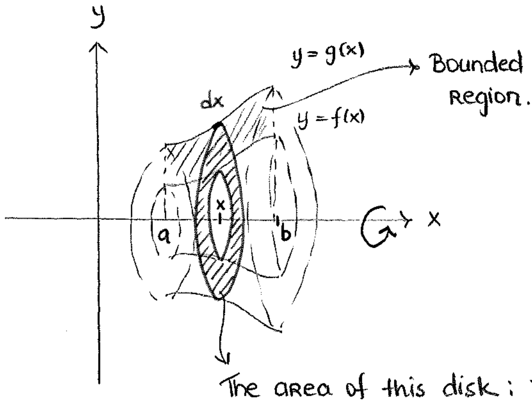
$$2x^2=-x^2+3x+6 \Rightarrow 3x^2-3x-6=0$$
$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=-1 \text{ and } x=2$$

The area is $A=\int_{-1}^{2}\left[(-x^2+3x+6)-(2x^2)\right]dx=\int_{-1}^{2}\left[-3x^2+3x+6\right]dx$

$$=\left[-\frac{\cancel{3}x^3}{\cancel{3}}+\frac{3x^2}{2}+6x\right]\Bigg|_{-1}^{2}=(-8+12+12)-(1+3-6)=16+2=18$$

Volumes of Solids of Revolution

(i) Rotate the bounded region about x-axis:



The area of this disk: $\pi\left[(g(x))^2-(f(x))^2\right]$

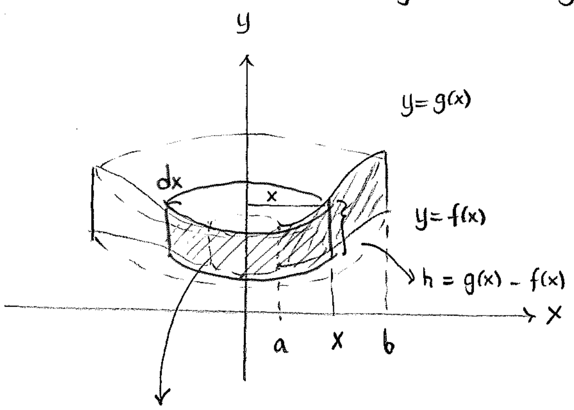
The volume of this slice: $\pi\left[(g(x))^2-(f(x))^2\right]dx$

The volume of the solid:

$$V=\int_a^b \pi\left[(g(x))^2-(f(x))^2\right]dx$$

(Disk Method)

(ii) Rotate the bounded region about y-axis:



The surface area of this shell: $2\pi x\left[g(x)-f(x)\right]$

The volume of this slice: $2\pi x\left[g(x)-f(x)\right]dx$

The volume of the solid:

$$V=\int_a^b 2\pi x\left[g(x)-f(x)\right]dx$$

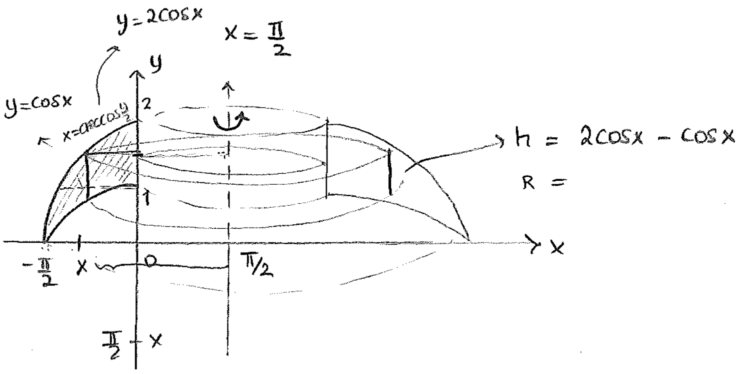
(Cylindrical Shell Method)

4. Let S be the solid obtained by rotating the finite region bounded by the curves $y = 2\cos x$, $y = \cos x$, $x = 0$ and $x = -\frac{\pi}{2}$ about the line $x = \frac{\pi}{2}$. Write down the integrals (without evaluating) giving the volume of S by using

(a) Slicing (disk method)

(b) Cylindrical shell method.

Solution:



(a) For an arbitrary $y \in [1,2]$: The area of the slice: $\pi\left[\left(\frac{\pi}{2} - \arccos\frac{y}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2\right]$

For an arbitrary $y \in [0,1]$: The area of the slice: $\pi\left[\left(\frac{\pi}{2} - \arccos\frac{y}{2}\right)^2 - \left(\frac{\pi}{2} - \arccos y\right)^2\right]$

The volume $V = \int_0^1 \pi\left[\left(\frac{\pi}{2} - \arccos\frac{y}{2}\right)^2 - \left(\frac{\pi}{2} - \arccos y\right)^2\right] dy + \int_1^2 \pi\left[\left(\frac{\pi}{2} - \arccos\frac{y}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2\right] dy$

(b) $V = \int_{-\frac{\pi}{2}}^{0} 2\pi\left(\frac{\pi}{2} - x\right)\left[2\cos x - \cos x\right] dx$

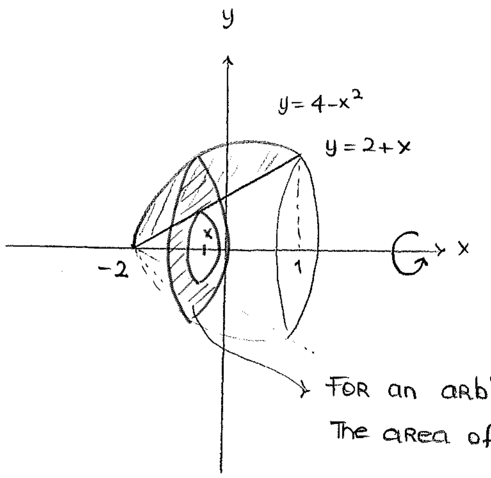
2. Let $f(x) = 4 - x^2$ and $g(x) = 2 + x$. Find the volume of a solid S obtained by rotating the region enclosed by f and g, about x-axis.

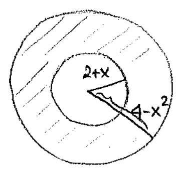
Solution:

The intersection of $y = 4 - x^2$ and $y = 2 + x$

$4 - x^2 = 2 + x \Rightarrow x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \Rightarrow x = -2$ & $x = 1$





For an arbitrary $x \in [-2, 1]$

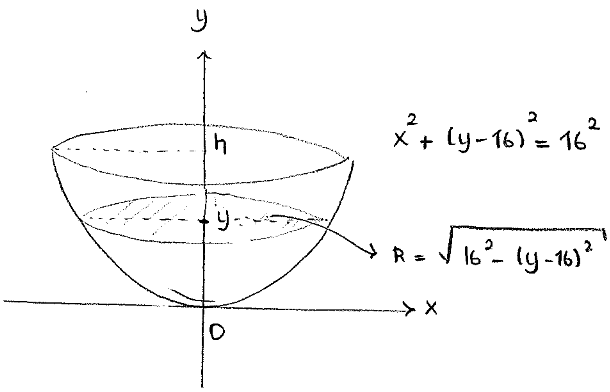
The area of this disk: $\pi\left[(4-x^2)^2 - (2+x)^2\right]$

Thus; the volume of S:

$$V(S) = \int_{-2}^{1} \pi\left[(4-x^2)^2 - (2+x)^2\right] dx \qquad \text{(by Disk method)}$$

3. Express the volume $V(h)$ of a water in 16 cm diameter hemispherical bowl as an integral and hence as a function of depth h of the water.

Solution:



For an arbitrary $y \in [0, h]$

The area of the disk : $\pi R^2 = \pi\left(16^2 - (y-16)^2\right)$

The volume is $\quad V = \int_0^h \pi\left(16^2 - (y-16)^2\right) dy$

MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 14

The Arc length of the Graph of a Function:

Let f be a function defined on $[a,b]$ and having a continuous derivative f' there. The arc length S of the curve $y=f(x)$ from $x=a$ to $x=b$ is given by

$$S=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

1. Find the arc lengths of the curves

(a) $y=\ln(\sec x)$, $0\leq x\leq \pi/4$

(b) $y^3=x^2$, $-1\leq x\leq 4$

Solution:

(a) $y=\ln(\sec x) \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\sec x}\cdot \sec x.\tan x=\tan x$

Thus; $$S=\int_0^{\pi/4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx=\int_0^{\pi/4}\sqrt{\frac{1}{\cos^2 x}}\,dx$$

$$=\int_0^{\pi/4}\sec x\,dx=\int_0^{\pi/4}\frac{\sec x\,(\sec x+\tan x)}{\sec x+\tan x}\,dx=\int_A^B \frac{1}{u}\,du=\ln|u|\Big|_A^B$$

$u=\sec x+\tan x \Rightarrow du=(\tan x.\sec x+\sec^2 x)dx$

$$=\ln|\sec x+\tan x|\Big|_0^{\pi/4}=\ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\underbrace{\ln|\sec 0+\tan 0|}_{0}=\ln\left(\frac{2}{\sqrt{2}}+1\right)$$

(b) $y=x^{2/3} \Rightarrow \dfrac{dy}{dx}=\dfrac{2}{3}x^{-1/3}$

$$S=\int_{-1}^{4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{-1}^{4}\sqrt{1+\frac{4}{9}x^{-2/3}}\,dx=\int_{-1}^{4}\frac{\sqrt{9x^{2/3}+4}}{3|x|^{1/3}}\,dx$$

$u=9x^{2/3}+4$
$du=9.\frac{2}{3}.x^{-1/3}dx$

$$=\frac{1}{18}\int_A^B\sqrt{u}\,du=\frac{1}{18}\frac{u^{3/2}}{\frac{3}{2}}\Big|_A^B=\frac{1}{18}\cdot\frac{2}{3}\left(9x^{2/3}+4\right)\Big|_{-1}^{4}$$

2. Set up, but do not evaluate, integrals for the arc lengths of the curves

(a) $y = 2^x$, $0 \leqslant x \leqslant 3$

(b) $y = x - y^3$, $1 \leqslant y \leqslant 4$

Solution:

(a) $y = 2^x \Rightarrow \dfrac{dy}{dx} = 2^x . \ln 2$

$$S = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + (2^x . \ln 2)^2}\, dx$$

(b) $y = x - y^3 \Rightarrow x = y^3 + y \Rightarrow \dfrac{dx}{dy} = 3y^2 + 1$

$$S = \int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^4 \sqrt{1 + (3y^2+1)^2}\, dy$$

Area of a Surface of Revolution:

(i) If $f'(x)$ is continuous on $[a,b]$ and the curve $y = f(x)$ is rotated about the x-axis, the area of surface of revolution is

$$S = 2\pi \int_{x=a}^{x=b} |y|\, ds = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\, dx$$

If rotation is about y-axis:

$$S = 2\pi \int_{x=a}^{x=b} |x|\, ds = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2}\, dx$$

(ii) If $g'(y)$ is continuous on $[c,d]$ and the curve $x = g(y)$ is rotated about the x-axis, the area of surface of revolution is

$$S = 2\pi \int_{y=c}^{y=d} |y|\, ds = 2\pi \int_c^d |y| \sqrt{1 + (g'(y))^2}\, dy$$

If rotation is about x-axis:

$$S = 2\pi \int_{y=c}^{y=d} |x|\, ds = 2\pi \int_c^d |g(y)| \sqrt{1 + (g'(y))^2}\, dy$$

4. Set up, but do not evaluate, integrals for the area of the surface obtained by rotating the curve about the given axis

(a) $y = \ln x$, $1 \leqslant x \leqslant 3$ x-axis and y-axis

(b) $y = \sin^2 x$, $0 \leqslant x \leqslant \pi/2$, x-axis

(c) $y = \sec x$, $1 \leqslant y \leqslant \pi/4$, y-axis

Solution:

(a) $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

Rotated about x-axis:

$$S = 2\pi \int_{x=1}^{x=3} |y|\, ds = 2\pi \int_{1}^{3} |\ln x| \sqrt{1 + \frac{1}{x^2}}\, dx$$

Rotated about y-axis:

$$S = 2\pi \int_{x=1}^{x=3} |x|\, ds = 2\pi \int_{1}^{3} x \cdot \sqrt{1 + \frac{1}{x^2}}\, dx$$

(b) $y = \sin^2 x \Rightarrow \dfrac{dy}{dx} = 2 \sin x . \cos x = \sin 2x$

$$S = 2\pi \int_{x=0}^{x=\frac{\pi}{2}} |y|\, ds = 2\pi \int_{0}^{\frac{\pi}{2}} |\sin^2 x| \sqrt{1 + \sin^2 2x}\, dx$$

3. Find the area of the surface obtained by Rotating the curve about the x-axis

(a) $9x = y^2 + 18$, $2 \leqslant x \leqslant 6$

(b) $y = \cos 2x$, $0 \leqslant x \leqslant \pi/6$

(c) $x = 1 + 2y^2$, $1 \leqslant y \leqslant 2$

(d) $y = e^{-x}$, $x \geqslant 0$

Solution:

(a) $x = \dfrac{y^2 + 18}{9}$ $\Rightarrow$ $x = 2 \Rightarrow y^2 + 18 = 18 \Rightarrow y = 0$ $\qquad 0 \leqslant y \leqslant 6$

$x = 6 \Rightarrow y^2 + 18 = 54 \Rightarrow y^2 = 36 \Rightarrow y = 6$

$$\frac{dx}{dy} = \frac{2y}{9}$$

$$S = 2\pi \int_{y=0}^{y=6} |y| \, ds = 2\pi \int_0^6 |y| \cdot \sqrt{1 + \frac{4}{81} y^2} \, dy$$

(b) $y = \cos 2x \Rightarrow \dfrac{dy}{dx} = -\sin(2x) \cdot 2$

$$S = 2\pi \int_{x=0}^{x=\pi/6} |y| \, ds = 2\pi \int_0^{\pi/6} |\cos 2x| \sqrt{1 + 4\sin^2(2x)} \, dx$$

(c) $x = 1 + 2y^2 \Rightarrow \dfrac{dx}{dy} = 4y$

$$S = 2\pi \int_{y=1}^{y=2} |y| \, ds = 2\pi \int_1^2 |y| \sqrt{1 + 16y^2} \, dy$$

(d) $y = e^{-x} \Rightarrow \dfrac{dy}{dx} = -e^{-x}$

$$S = 2\pi \int_{x=0}^{\infty} |y| \, ds = 2\pi \int_0^{\infty} |e^{-x}| \cdot \sqrt{1 + e^{-2x}} \, dx$$