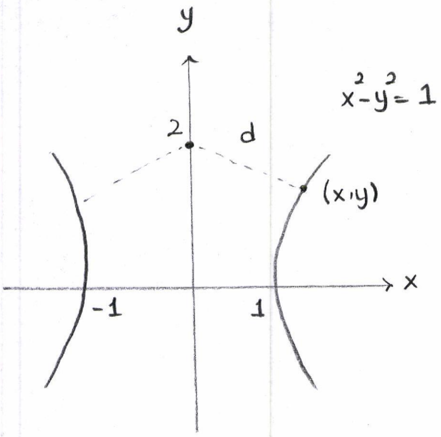
MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 8

1. Find all points on the curve $x^2 - y^2 = 1$ closest to the point $(x,y) = (0,2)$.

Solution:



The distance between an arbitrary point (x,y) on the curve and $(0,2)$:

$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

Since the point (x,y) is on $x^2 - y^2 = 1$

$$x^2 = 1 + y^2 \Rightarrow x = \sqrt{1+y^2} \quad \text{OR} \quad x = -\sqrt{1+y^2}$$

Since the graph is symmetric with respect to y-axis and the point is on y-axis, find the closest point on the first quadrant. Then we can represent x as $\sqrt{1+y^2}$.

Thus; $d(y) = \sqrt{(\sqrt{1+y^2})^2 + (y-2)^2} = \sqrt{1+y^2+y^2-4y+4} = \sqrt{2y^2-4y+5}$

We are trying to find absolute minimum value of d, for simplicity find the abs. minimum value of $f(y) = [d(y)]^2 = 2y^2 - 4y + 5$

$f'(y) = 4y - 4 \qquad f'(y) = 0 \Rightarrow 4y = 4 \Rightarrow y = 1.$

y	0		1	
f'		$-$		$+$
f		$\searrow$		$\nearrow$
			absolute minimum	

$f(1) = 2 - 4 + 5 = 3$ so $d(1) = \sqrt{3}$ is the minimum distance.

$x^2 = 1 + 1 \Rightarrow x = \sqrt{2}$

The point is $(\sqrt{2}, 1)$.

Moreover, symmetry with respect to y-axis is $(-\sqrt{2}, 1)$.

2. Find two numbers whose product is -12 and the sum of whose squares is a minimum.

Solution:

Let $x, y \in \mathbb{R}$. We are given $x \cdot y = -12$.

We need to find the absolute minimum value of $x^2 + y^2$

$x \cdot y = -12 \Rightarrow y = -\frac{12}{x}$

Define $f(x) = x^2 + \left(-\frac{12}{x}\right)^2 = x^2 + \frac{144}{x^2}$

$f'(x) = 2x - \frac{288}{x^3} = \frac{2x^4 - 288}{x^3} = \frac{2(x^4 - 12^2)}{x^3} = \frac{2(x^2+12)(x-\sqrt{12})(x+\sqrt{12})}{x^3}$

$f'(x) = 0 \Rightarrow x = \pm\sqrt{12}$ OR $x = 0$

x	$-\sqrt{12}$	0	$\sqrt{12}$	
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Local min   Local min

$f(-\sqrt{12}) = 12 + \frac{144}{12} = 24$

$f(\sqrt{12}) = 12 + \frac{144}{12} = 24$

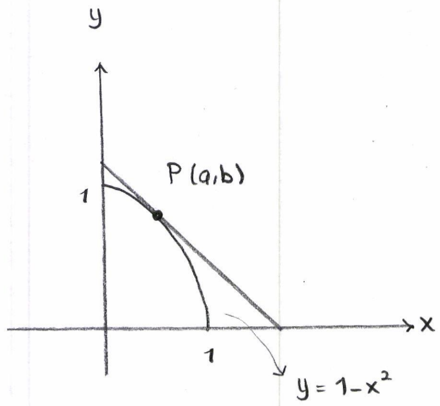
$x = -\sqrt{12} \Rightarrow (-\sqrt{12}) \cdot y = -12 \Rightarrow y = \sqrt{12}$

(OR $x = \sqrt{12} \Rightarrow y = -\sqrt{12}$)

Thus; the numbers are $\sqrt{12}$ & $-\sqrt{12}$.

3. Find the x-coordinate of the point P on the parabola $y = 1 - x^2$; $0 < x \leq 1$ where the triangle that is enclosed by the tangent line at P and the coordinate axes has the smallest area.

Solution:



The equation of the tangent line :

$$y'(x) = -2x \Rightarrow y'(a) = -2a$$

$$y - b = -2a(x-a)$$

$$y + 2ax = 2a^2 + b$$

$$x = 0 \Rightarrow y = 2a^2 + b$$

$$y = 0 \Rightarrow x = \frac{2a^2+b}{2a}$$

Thus; the area of the triangle is $\frac{1}{2} \cdot (2a^2+b) \cdot \frac{2a^2+b}{2a}$

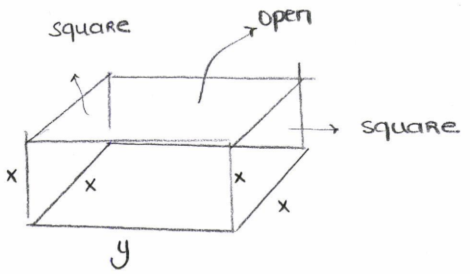
The point P is on the parabola ; so we have $b = 1 - a^2$

Define $f(a) = \frac{1}{4} \frac{(2a^2+1-a^2)^2}{a} = \frac{1}{4a} (a^2+1)^2$ find the absolute minimum of f

$$f'(a) = \frac{2(a^2+1) \cdot 2a \cdot \cancel{4}a + 4(a^4+2a^2+1)}{\cancel{4}a^2} = \frac{4a^4 + 4a^2 + a^4 + 2a^2 + 1}{a^2}$$

4. A Rectangular container with two square sides and open top is to have a volume $V = 108\ m^3$. Find the dimensions of the container with minimum surface area.

Solution:



The volume is $V = y \cdot x^2 \Rightarrow 108 = y \cdot x^2 \Rightarrow y = \frac{108}{x^2}$

The surface area is

$3xy + 2x^2$

$f(x) = 3x \cdot \frac{108}{x^2} + 2x^2$ find the absolute minimum.

$f(x) = \frac{3.108}{x^2} + 2x^2 \Rightarrow f'(x) = -\frac{6.108}{x^3} + 4x \Rightarrow \frac{-6.108 + 4x^4}{x^3} = \frac{4\,(x^2 + \sqrt{162})\,(x^2 - \sqrt{162})}{x^3}$

x		0	$162^{1/4}$	
f'	$+$	$-$		$+$
f	$\nearrow$	$\searrow$	min	$\nearrow$

$x = \sqrt[4]{162}$

Sketching Graph

Asymptotes

(i) Vertical Asymptote: The graph $y = f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \to a^-} f(x) = \pm\infty$ OR $\lim_{x \to a^+} f(x) = \pm\infty$ OR both.

(ii) Horizontal Asymptote: The graph $y = f(x)$ has horizontal asymptote at $y = L$ if either $\lim_{x \to \infty} f(x) = L$ OR $\lim_{x \to -\infty} f(x) = L$ OR both.

(iii) Oblique Asymptote; $y = ax + b$ is an asymptote of $y = f(x)$ if either $\lim_{x \to -\infty} [f(x) - (ax+b)] = 0$ OR $\lim_{x \to \infty} [f(x) - (ax+b)] = 0$.

Sketching Graph

(i) f: - Domain
- Intercepts and some point
- The symmetry of graph
- Asymptotes

(ii) f': - Intervals of increase and decrease
- Local extreme values

(iii) f'': - Concavity
- Inflection points.

5. Sketch the graph of

(a) $f(x) = \dfrac{x^2+2x}{x^2-1}$

(b) $f(x) = x^2 . e^{1/x}$

Solution:

(a) (i) Informations from f:

$x^2-1=0 \Rightarrow x=\pm 1$ Domain of f $\mathbb{R}-\{-1,1\}$

$x=0 \Rightarrow y=0$ So $(0,0)$ and $(-2,0)$ are intercepts.

$x=-2 \Rightarrow y=0$

(ii) Informations from f':

$$f'(x) = \frac{(2x+2)(x^2-1)-2x(x^2+2x)}{(x^2-1)^2} = -\frac{2(x^2+x+1)}{(x^2-1)^2} \Rightarrow f'(x)<0 \text{ for all } x\in\mathbb{R}-\{-1,1\}$$

f is decreasing.

(iii) Informations from f'':

$$f''(x) = \frac{2(2x^3+3x^2+6x+1)}{(-1+x^2)^3} = 0 \Rightarrow x=-0.18, \quad x=\pm 1$$

x		-1	-0.18	1
f'	$-$	$-$	$-$	$-$
f	↘	↘	↘	↘
f''	$-$	$+$	$-$	$+$
f	∩	∪	∩	∪

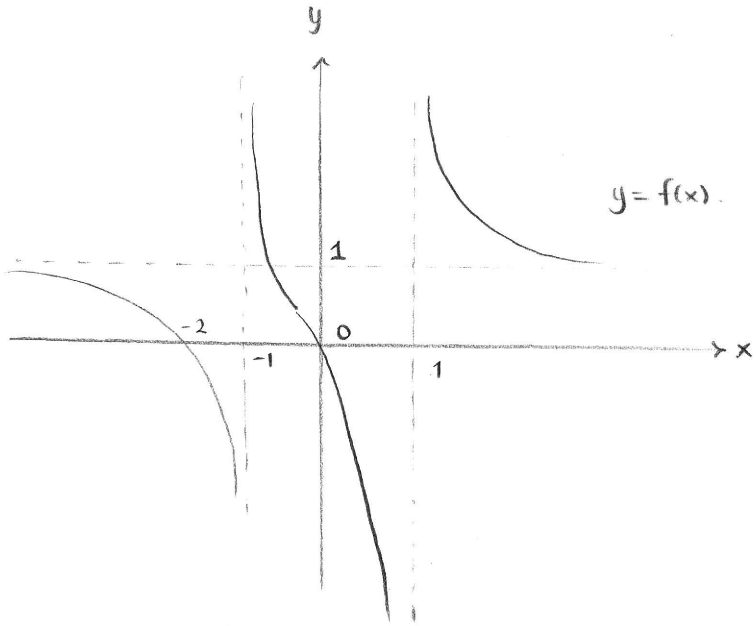
Asymtotes:

$$\lim_{x\to -1^-} \frac{x^2+2x}{x^2-1} = -\infty \qquad \lim_{x\to -1^+} \frac{x^2+2x}{x^2-1} = \infty$$

$$\lim_{x\to 1^-} \frac{x^2+2x}{x^2-1} = -\infty \qquad \lim_{x\to 1^+} \frac{x^2+2x}{x^2-1} = \infty$$

Thus; $x=1$ and $x=-1$ are vertical asymptotes.

$$\left.\begin{aligned} \lim_{x\to \infty} \frac{x^2+2x}{x^2-1} &= 1 \\ \lim_{x\to -\infty} \frac{x^2+2x}{x^2-1} &= 1 \end{aligned}\right\} \; y=1 \text{ is horizontal asymptote}$$


y
1
-2
0
-1
1
x
y = f(x).