MATH 119 – CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 7

Indeterminate Forms: $[0/0]$, $[\infty/\infty]$, $[0.\infty]$, $[\infty-\infty]$, $[0^0]$, $[\infty^0]$, $[1^\infty]$

L'Hôpital's Rule: Suppose f and g are differentiable on (a,b) and $g'(x) \neq 0$. Suppose also

(i) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$

(ii) $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L$ (L can be finite or $\pm\infty$)

Then, $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = L$

! $x\to a^+$ can be replaced by $x\to b^-$ or for $a<c<b$, $x\to c$. Moreover; a can be $-\infty$, b can be $+\infty$.

1. Evaluate the following limits

(a) $\lim\limits_{x\to\infty} (\pi - 2\arctan x)\ln x$

(b) $\lim\limits_{x\to\infty} \left[(x+2)e^{1/x} - x\right]$

(c) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$

(d) $\lim\limits_{x\to -1} \dfrac{x.\ln|x| - x - 1}{(x+1)^2}$

(e) $\lim\limits_{x\to\infty} \left(1 + \dfrac{2}{x} - \dfrac{3}{x^2}\right)^x$

(f) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$

Solution:

(a) $\lim\limits_{x\to\infty} (\pi - 2\arctan x).\ln x \overset{[0.\infty]}{=} \lim\limits_{x\to\infty} \dfrac{\pi - 2\arctan x}{\frac{1}{\ln x}} \overset{\left[\frac{0}{0}\right]}{=}$

Hatalı.

$\overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{-2\frac{1}{1+x^2}}{\frac{1}{\frac{1}{x}}} = \lim\limits_{x\to\infty} \left[-2 \cdot \dfrac{1}{x+x^3}\right] = 0$

(b) $\lim\limits_{x\to\infty}\left[(x+2)e^{1/x}-x\right] \overset{[\infty-\infty]}{=} \lim\limits_{x\to\infty}\left[x.e^{1/x}+2e^{1/x}-x\right] = \lim\limits_{x\to\infty}\left[\underbrace{x\left(e^{1/x}-1\right)}_{(*)}+\underbrace{2e^{1/x}}_{(**)}\right]$

$= 1+2 = 3$

(*) $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{\frac{1}{x}} \quad \left[\frac{0}{0}\right] \underset{L'H}{} \quad \lim\limits_{x\to\infty}\dfrac{e^{1/x}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 1$

(**) $\lim\limits_{x\to\infty} 2e^{1/x} = 2$

(c) $\lim\limits_{x\to1^+}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right) \overset{[\infty-\infty]}{=} \lim\limits_{x\to1^+}\left[\dfrac{x-1-\ln x}{\ln x.(x-1)}\right] \underset{L'H}{\left[\frac{0}{0}\right]} = \lim\limits_{x\to1^+}\dfrac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$

$= \lim\limits_{x\to1^+}\dfrac{\frac{x-1}{x}}{\frac{x-1+x\ln x}{x}} = \lim\limits_{x\to1^+}\dfrac{x-1}{x-1+x.\ln x} \underset{L'H}{\left[\frac{0}{0}\right]} = \lim\limits_{x\to1^+}\dfrac{1}{1+\ln x+x.\frac{1}{x}} = \dfrac{1}{2}$

(e) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}-\dfrac{3}{x^2}\right)^x \quad [0^\infty]$

$y = \lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}-\dfrac{3}{x^2}\right)^x \Rightarrow \ln y = \ln\left[\lim\limits_{x\to\infty}\underbrace{\left(1+\dfrac{2}{x}-\dfrac{3}{x^2}\right)^x}_{\text{is continuous}}\right]$

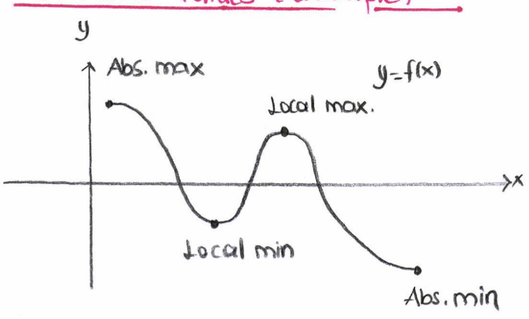
$\ln y = \lim\limits_{x\to\infty}\left[\ln\left(1+\dfrac{2}{x}-\dfrac{3}{x^2}\right)^x\right] = \lim\limits_{x\to\infty} x.\ln\left(1+\dfrac{2}{x}-\dfrac{3}{x^2}\right) \overset{[0.\infty]}{=} \lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{x}} \underset{L'H}{\left[\frac{0}{0}\right]}$

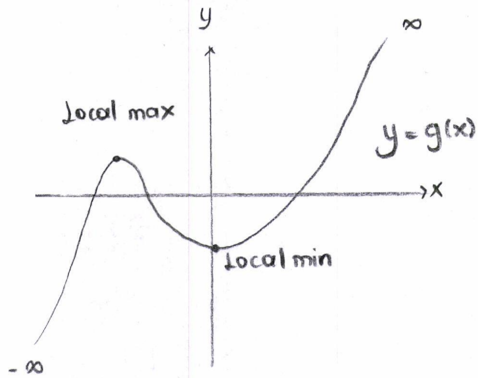
$= \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+\frac{2}{x}-\frac{3}{x^2}}\cdot\left(-\frac{2}{x^2}+\frac{6}{x^3}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x\to\infty}\dfrac{1}{1+\cancel{\frac{2}{x}}-\cancel{\frac{3}{x^2}}}\left(2-\cancel{\frac{6}{x}}\right) = 2$

$\ln y = 2 \Rightarrow y = e^2$

(f) $\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3} \underset{L'H}{\left[\frac{0}{0}\right]} \lim\limits_{x\to0}\dfrac{1-\cos x}{3x^2} \underset{L'H}{\left[\frac{0}{0}\right]} \lim\limits_{x\to0}\dfrac{\sin x}{6x} \underset{L'H}{\left[\frac{0}{0}\right]} \lim\limits_{x\to0}\dfrac{\cos x}{6} = \dfrac{1}{6}$

Extreme Values (Example)





No abs max & min.

Extreme Values

Theorem: If the domain of f is closed and finite interval and f is continuous, then f must have an absolute maximum and absolute minimum value.

A function f can have local extreme values only at

(i) Critical Points of f $(x \in D(f)$ and $f'(x)=0)$

(ii) Singular Points of f $(x \in D(f)$ and $f'(x)$ does not exist$)$

(iii) End Points of the domain of f.

2. Find the absolute maximum and minimum values of $f(x)=\dfrac{x^2+4}{x}$ on the following intervals if they exist:

(a) $[1,4]$ (b) $[-1,3]$ (c) $(-\infty,-2]$ (d) $(-2,\infty)$ (e) $(-\infty,\infty)$

Solution:

(a) The function $f(x)=\dfrac{x^2+4}{x}$ is continuous on the closed and bounded interval $[1,4]$.

Thus; f has absolute maximum and minimum values on $[1,4]$.

To find them, find special points of f:

$$f'(x)=\frac{2x.x-x^2-4}{x^2}=\frac{x^2-4}{x^2}=\frac{(x-2)(x+2)}{x^2}$$

$x=0 \notin D(f)$

$x=2 \in D(f)$ critical point

$x=-2 \notin D(f)$

$f'(x)$ is defined for all $x \in [1,4]$ so there is no singular point.

$x=1$ and $x=4$ are end points of the domain.

x	1	2	4
f'	$-$	$+$	
f	↘	↗	

Local minimum

When we compare the values;

$f(1)=\dfrac{1+4}{1}=5$

$f(2)=\dfrac{4+4}{2}=\dfrac{8}{2}=4$

$f(4)=\dfrac{16+4}{4}=\dfrac{20}{4}=5$

$f(1)$ & $f(4)$ absolute maximum value.

$f(2)$ absolute minimum value.

(b) $f(x) = \dfrac{x^2+4}{x}$ is not defined at $x = 0$.

$$\lim_{x \to 0^-} \frac{x^2+4}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{x^2+4}{x} = \infty$$

It is not possible to find values of f greater than ∞ OR less than $-\infty$ so it has no absolute maximum OR absolute minimum on $[-1,3]$.

(It may have local minimum and local maximum values.)

(c) Since the interval $(-\infty,-2]$ is open we need to find limit when $x \to -\infty$

$$\lim_{x \to -\infty} \frac{x^2+4}{x} \overset{L'H}{=} \lim_{x \to -\infty} \frac{2x}{1} = -\infty$$

f is continuous on $(-\infty,-2]$, it is impossible to find a value of f less than $-\infty$ so f has no absolute minimum. But it has absolute maximum value.

$f'(x) = 0$ when $x = -2$ is critical point and end point of the domain.
$x = 0 \notin D(f)$
$x = 2 \notin D(f)$

There is no singular point on $(-\infty,-2]$.

x	$-\infty$ -2	
f'	$+ + + +$	
f	$\nearrow \nearrow \nearrow \nearrow$	

Thus; $f(-2) = \dfrac{4+4}{-2} = \dfrac{8}{-2} = -4$ is the absolute maximum value.

(d) Exercise.

3. Determine whether the given functions has any local or absolute extreme values; and find those values if possible.

(a) $f(x)=|x^2-x-6|$ on $[-3,3]$

(b) $f(x)=|x^2-x-6|$

(c) $f(x)=\dfrac{x}{\sqrt{x^4+1}}$

(d) $f(x)=x-2\arctan x$

Solution:

(a) $x^2-x-6=(x-3)(x+2)=0$ $\quad x=3$, $x=-2$

x	-3	-2	3
x^2-x-6	$+$	$-$	$+$

$$f(x)=\begin{cases} x^2-x-6 & \text{if } -3\leqslant x\leqslant -2 \\ -x^2+x+6 & \text{if } -2< x\leqslant 3 \end{cases}$$

You can find the abs max and min values for each part of the piecewisely defined f

But shortly; $f'(x)=(2x-1).\ \mathrm{sgn}(x^2-x-6)$

At $x=-2$ and $x=3$, $f'(x)$ is not defined, they are singular points.

$2x-1=0 \Rightarrow x=1/2$ is critical point.

x	-3	-2	$1/2$	3	
$2x-1$	$-$	$-$	$-$	$+$	$+$
sgn	$+$	$+$	$-$	$-$	$+$
f'		$-$ ↘	$+$ ↗	$-$ ↘	
		local min	local max		

$x=3$ and $x=-3$ are end points.

$f(-2)=|4+2-6|=0$ } absolute min.
$f(3)=|9-3-6|=0$

$f(1/2)=\left|\frac{1}{4}-\frac{1}{2}-6\right|=\left|\frac{1-2-24}{4}\right|=\frac{25}{4}$ → absolute max.

$f(-3)=|9+3-6|=6$

(b) Exercise.

(c) $f(x) = \dfrac{x}{\sqrt{x^4+1}}$ the domain of f is $\mathbb{R}$.

$$f'(x) = \frac{\sqrt{x^4+1} - x \cdot \dfrac{\overset{2}{\cancel{4}}x^3}{\cancel{2}\sqrt{x^4+1}}}{x^4+1} = \frac{x^4+1-2x^4}{(x^4+1)^{3/2}} = \frac{1-x^4}{(x^4+1)^{3/2}}$$

$f'(x)=0 \Rightarrow x = \pm 1 \in \mathbb{R}$ are critical points.

There is no singular point and no end point.

$\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^4+1}} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^4+1}} = 0$

x		-1		1	
f'	−		+		−
f	0 ↘	Abs.min	↗	Abs.max	↘ 0

$f(-1) = \dfrac{-1}{\sqrt{2}}$ absolute minimum

$f(1) = \dfrac{1}{\sqrt{2}}$ absolute maximum

(d) $f(x) = x - 2\arctan x$, the domain of f is $\mathbb{R}$.

$$f'(x) = 1 - \frac{2}{1+x^2} = \frac{1+x^2-2}{1+x^2} = \frac{x^2-1}{1+x^2} = \frac{(x-1)(x+1)}{1+x^2}$$

$f'(x) = 0 \Rightarrow x = \pm 1 \in \mathbb{R}$ are critical points.

There is no singular point and no end point.

$\lim\limits_{x\to\infty} (x - 2\arctan x) = \infty$, $\lim\limits_{x\to-\infty} (x - 2\arctan x) = -\infty$

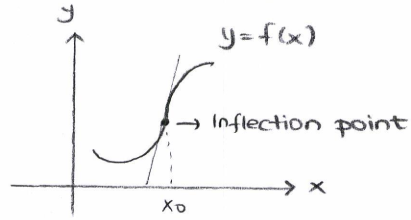
x		-1		1	
f'	+		−		+
f	$-\infty$ ↗	local max	↘	local min	↗ ∞

$f(-1) = -1 - 2\arctan(-1) = -1 + \frac{\pi}{2}$

$f(1) = 1 - 2\arctan 1 = 1 - \frac{\pi}{2}$

Concavity and Inflections

f is concave up on an open interval if f' is increasing and concave down if f' is decreasing.



$(x_0, f(x_0))$ is an inflection point of $y=f(x)$ if

(i) $y=f(x)$ has a tangent line at $x=x_0$.

(ii) Concavity of f is opposite on opposite sides of x_0.

4. Find and classify all local extreme values of $f(x)$. Determine whether any of these extreme values are absolute. Find the intervals on which $f(x)$ is increasing, decreasing, concave up and concave down. Locate any inflection points if exist.

(a) $f(x) = x\sqrt{4-x^2}$

(b) $f(x) = (x^2-x-2).e^x$

(c) $f(x) = \dfrac{x}{\ln x}$

(d) $f(x) = x + \sin x$

Solution:

(a) $4-x^2 = (2-x)(2+x) = 0 \qquad x=2 \quad x=-2$

x		-2		2	
$4-x^2$	$-$		$+$		$-$

Domain of f is $[-2,2]$

$$f'(x) = \sqrt{4-x^2} + x\ \frac{1}{2}\cdot\frac{1}{\sqrt{4-x^2}}(-\cancel{2}x) = \frac{4-x^2-x^2}{\sqrt{4-x^2}} = \frac{2\,(2-x^2)}{\sqrt{4-x^2}}$$

$x=2$ & $x=-2$ singular points & end points

$x=\pm\sqrt{2}$ are critical points

$$f''(x) = \frac{2.(-2x)\sqrt{4-x^2} - \frac{1}{2}\frac{(-2x)}{\sqrt{4-x^2}}\,2.(2-x^2)}{4-x^2} = \frac{2x\,(x^2-6)}{(4-x^2)^{3/2}}$$

$x=0 \rightarrow$ inflection point

$x=\pm\sqrt{6} \notin D(f)$

x	-2		$-\sqrt{2}$		0		$\sqrt{2}$		2		$\sqrt{6}$	
f'		$-$		$+$		$+$		$-$				
f		$\searrow$		$\nearrow$		$\nearrow$		$\searrow$				
f''		$+$		$+$		$-$		$-$				$+$
f		$\cup$		$\cup$		$\cap$		$\cap$				

$\cup$ local min $\cap$ local max

inflection point

$f(\sqrt{2}) = \sqrt{2}\sqrt{4-2} = 2 \rightarrow$ abs max

$f(-\sqrt{2}) = -\sqrt{2}\sqrt{4-2} = -2 \rightarrow$ abs. min

$f(2) = 0$

$f(-2) = 0$

f is increasing on $[-\sqrt{2},\sqrt{2}]$, decreasing on $[-2,-\sqrt{2}]$ $[\sqrt{2},2]$

Concave up on $[-2,0]$ down on $[0,2]$

(b) $f(x) = (x^2 - x - 2)\, e^x$ domain is $\mathbb{R}$

$f'(x) = (2x-1) \cdot e^x + e^x (x^2 - x - 2)$

$= e^x \, (2x - 1 + x^2 - x - 2) = e^x (x^2 + x - 3)$ $\qquad \Delta = 1 - 4(-3) \cdot 1$