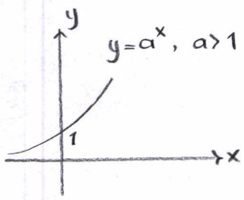
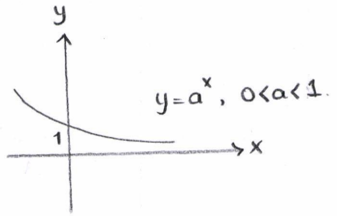
MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 6

Exponential Function: $f: \mathbb{R} \to (0,\infty)$, $f(x)=a^x$ for $a>0$.



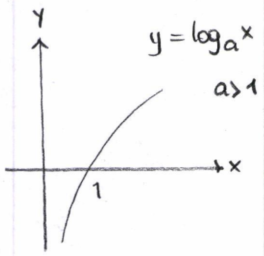


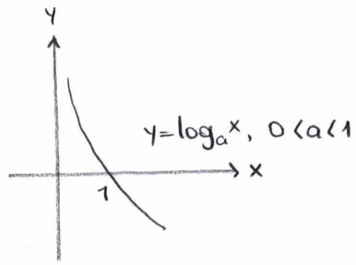
$f(x)=e^x$ is the natural exponential

$\frac{d}{dx} a^x = a^x . \ln a$, $\frac{d}{dx} e^x = e^x$

Logarithmic Function: For $a>0$, $a\neq 1$ $f(x)=\log_a x$ is the inverse of a^x

$y=\log_a x \Leftrightarrow x=a^y$





$f(x)=\ln x$ is the natural logarithm.

$\frac{d}{dx} \log_a x = \frac{1}{x.\ln a}$, $\frac{d}{dx}\ln x = \frac{1}{x}$

1. Prove the following inequalities.

(a) $\ln(1+x) < x$ for all $x>0$.

(b) $\cos x > 1-\frac{x^2}{2}$ for all $x>0$.

Solution:

(a) Define $f(x)=\ln(1+x)-x$ for $x>0$. f is a continuous function.

$f'(x)=\frac{1}{1+x}-1=\frac{1-1-x}{1+x}=-\frac{x}{1+x}$ $\quad x=0$, $x=-1$ (not in the domain)

x	0	
f'		$- - - -$
f		$\searrow \searrow \searrow$

f is strictly decreasing and $f(0)=\ln 1-0=0$, we have f is negative for all $x>0$.

$f(x)<0$ for $x>0$ that proves the inequality.

$\ln(1+x)-x<0 \Rightarrow \ln(1+x)<x$ for all $x>0$.

(b) Exercise.

2. Find the intervals on which f is increasing and decreasing where $f(x) = x.e^{-x}$.

Solution:

$f'(x) = e^{-x} + x.(-e^{-x}) = e^{-x}(1-x)$ $f'(x) = 0 \Rightarrow 1-x=0 \Rightarrow x=1$.

x		1	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

f is increasing on $(-\infty, 1)$

f is decreasing on $(1, \infty)$

3. Find $\frac{dy}{dx}$ at $x=0$ if the differentiable function $y=y(x)$ is given implicitly by

$3xe^{2y} + y.e^{x} = 4e^{x} - 3$.

Solution:

To find $y(0)$: $3.0.e^{2y(0)} + y(0).e^{0} = 4.e^{0} - 3 \Rightarrow y(0) = 4-3 = 1 \Rightarrow y(0) = 1$.

Differentiate $3.xe^{2y} + ye^{x} = 4e^{x} - 3$ implicitly assuming $y = y(x)$.

$3.e^{2y} + 3x.e^{2y}.2y' + y'.e^{x} + y.e^{x} = 4e^{x}$ Let $x=0$ and $y(0)=1$.

$3.e^{2} + 3.0.e^{2}2y'(0) + y'(0).e^{0} + 1.e^{0} = 4.e^{0} \Rightarrow y'(0) + 1 + 3e^{2} = 4$

$y'(0) = 3 - 3e^{2}$

4. Show that $f(x) = x^5 + \arctan x + e^{x} + 119$ has an inverse defined on the Range of f and find $(f^{-1})'(120)$.

Solution:

$f'(x) = 5x^4 + \frac{1}{1+x^2} + e^{x}$ for all $x \in \mathbb{R}$ $f'(x) > 0$ so f is strictly increasing. Hence;

f is one-to-one function. So f is invertible.

$$(f^{-1})'(120) = \frac{1}{f'(f^{-1}(120))} = \frac{1}{f'(0)} = \frac{1}{5.0 + \frac{1}{1+0^2} + e^{0}} = \frac{1}{2}$$

$f^{-1}(120) = x \Rightarrow f(x) = 120$

$x = 0 \Rightarrow 0 + \arctan 0 + e^{0} + 119 = 120$

5. Find y' if

(a) $y = x^{a^2} + a^{x^2} + 2^{a^x}$ for some constant $a>0$.

(b) $y = \ln(\ln x) + e^{e^x}$

(c) $y = x^{\ln x}$

Solution:

(a) $y' = a^2 . x^{(a^2-1)} + a^{x^2} . \ln a . (2x) + 2^{a^x} . \ln 2 . a^x . \ln a$

(b) $y' = \frac{1}{\ln x} . \frac{1}{x} + e^{e^x} . e^x$

(c) Logarithmic Differentiation: To take the derivative of $y = [f(x)]^{g(x)}$ take the natural logarithm of both sides.

$\ln y = \ln [f(x)]^{g(x)} \Rightarrow \ln y = g(x) . \ln [f(x)]$

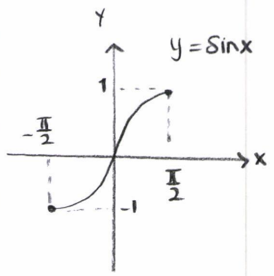
$\frac{y'}{y} = g'(x) . \ln [f(x)] + g(x) . \frac{f'(x)}{f(x)}$

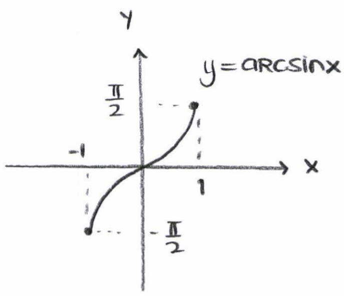
$y = x^{\ln x} \Rightarrow \ln y = \ln [x^{\ln x}] = \ln x . \ln x = (\ln x)^2$

$\frac{y'}{y} = 2 . \ln x . \frac{1}{x} \Rightarrow y' = x^{\ln x} . 2 . \ln x . \frac{1}{x}$

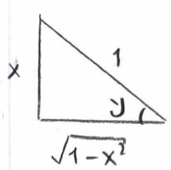
Inverse Trigonometric Functions.

Arcsine: $\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ (one-to-one)



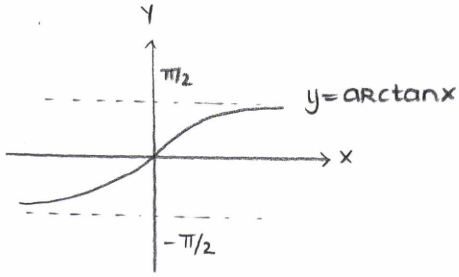


$y = \arcsin x \Rightarrow \sin y = x \Rightarrow \cos y . y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$



$\cos y = \sqrt{1-x^2}$

Arctan: $\mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Arccos: $[-1,1] \to [0,\pi]$ $\qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$

6. Find $\frac{dy}{dx}$ if

(a) $y = e^{-(\arcsin x)^2}$

(b) $y = 2x \cdot \arctan(x^2+1)$

(c) $y = \arctan(\arccos\sqrt{x})$

Solution:

(a) $\frac{dy}{dx} = e^{-(\arcsin x)^2} \cdot \left[-2 \cdot \arcsin x \cdot \frac{1}{\sqrt{1-x^2}}\right]$

(b) $\frac{dy}{dx} = 2 \cdot \arctan(x^2+1) + 2x \cdot \frac{1}{1+(x^2+1)^2} \cdot 2x$

(c) $\frac{dy}{dx} = \frac{1}{1+(\arccos\sqrt{x})^2} \cdot \left[-\frac{1}{\sqrt{1-x}}\right] \cdot \frac{1}{2\sqrt{x}}$

7. Verify that the point $P_0\,(1, \frac{1}{2})$ is on the curve C defined implicitly by the equation $\arcsin(xy) = x^2y + 5x - \frac{11}{2} + \frac{\pi}{6}$. Then find the tangent line to C at $P_0\,(1, \frac{1}{2})$.

Solution:

Check $P_0\,(1, \frac{1}{2})$ is on the curve

$$\arcsin \frac{1}{2} = \frac{1}{2} + 5 - \frac{11}{2} + \frac{\pi}{6} \Rightarrow \frac{\pi}{6} = \frac{\pi}{6} \checkmark \quad \Rightarrow y(1) = \frac{1}{2}$$

Differentiate $\arcsin(xy) = x^2y + 5x - \frac{11}{2} + \frac{\pi}{6}$ assuming $y = y(x)$

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot (y + xy') = 2xy + x^2y' + 5 \qquad \text{let } x = 1, \; y = \frac{1}{2}$$

$$\frac{1}{\sqrt{1-\frac{1}{4}}} \left(\frac{1}{2} + y'(1)\right) = 2 \cdot \frac{1}{2} + y'(1) + 5 \Rightarrow \frac{2}{\sqrt{3}} \left(\frac{1}{2} + y'(1)\right) = 6 + y'(1) \; \ldots \quad y'(1) = m$$

The tangent line equation is $y - \frac{1}{2} = m\,(x-1)$.

8. Show that the equation $4\arctan(x) = 3 - 2x - x^3$ has exactly one solution on $x \in (0,1)$.

Solution:

Define $f(x) = 4\arctan x + x^3 + 2x - 3$ on $(0,1)$ f is continuous.

$f(0) = 4\arctan 0 + 0 + 0 - 3 = -3 < 0$

$f(1) = 4\arctan 1 + 1 + 2 - 3 = 4 \cdot \frac{\pi}{4} > 0$

f is continuous on $[0,1]$ and 0 is between $f(0)$ & $f(1)$ by the Inter-mediate Value Theorem; there exists $c \in [0,1]$ such that $f(c) = 0$.

Assume that there exists $c_1 \in (0,1)$ such that $f(c_1) = 0$ and $c_1 > c$

f is continuous on $[c, c_1]$ and differentiable on (c, c_1) by the Mean-Value Theorem there exists $k \in (c, c_1)$ such that

$$f'(k) = \frac{f(c_1) - f(c)}{c_1 - c} = 0$$

$$f'(x) = \frac{4}{1+x^2} + 3x^2 + 2 > 0$$ for all $x \in (0;1)$ so there is no root.

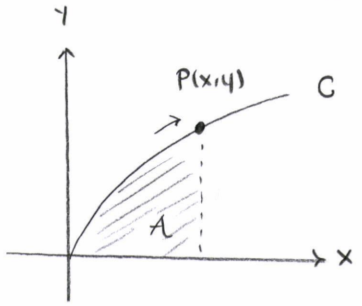
The claim is wrong. $\lightning$

Thus; $4\arctan x = 3 - 2x - x^3$ has exactly one real root on $(0,1)$.

9. The shaded area $A(t)$ in the figure given by the formula

$$A = \frac{2}{3}(x+1)(y+1) - x - \frac{2}{3}$$

denotes an increasing function of time t, as the point $P(x,y)$ moves ahead along the curve C. Find the rate of change of A if x and y coordinates of the point $P(x,y)$ are both increasing at a rate of 1 cm/min when $A = \frac{11}{3}$ cm^2 and $y = 2$ cm.



Solution:

When $A = \frac{11}{3}$ and $y = 2$

$$\frac{11}{3} = \frac{2}{\cancel{3}}(x+1)\cdot\cancel{3} - x - \frac{2}{3} \Rightarrow 2x + 2 - x = \frac{13}{3} \Rightarrow x = \frac{13}{3} - 2 = \frac{13-6}{3} = \frac{7}{3}$$

$$A = \frac{2}{3}(xy + x + y + 1) - x - \frac{2}{3} = \frac{2}{3}(xy + x + y) - x$$

$$\frac{dA}{dt} = \frac{2}{3}\left(\frac{dx}{dt}\cdot y + \frac{dy}{dt}\cdot x + \frac{dx}{dt} + \frac{dy}{dt}\right) - \frac{dx}{dt}$$

$$= \frac{2}{3}\left(1.2 + 1.\frac{7}{3} + 1 + 1\right) - 1$$